Math 245 - Fall 2019 HW#1 - Solutions

1.) $\frac{dy}{dx} = 1 + 2x + y^2 + 2xy^2 \Rightarrow \frac{dy}{dx} = (1+2x)[1+y^2] \Rightarrow$

$\int \frac{dy}{1+y^2} = \int (1+2x)\,dx \Rightarrow \arctan(y) = x + x^2 + c_1$

(separable d.e.) $\Rightarrow \boxed{y = \tan(x + x^2 + c_1)}$

2.) $y' = -\frac{1}{t^2} - \frac{1}{t}y + y^2 \longrightarrow$ Riccati eqn.

$y_1(t) = \frac{1}{t}$ is a soln. $\Rightarrow y(t) = \frac{1}{t} + \frac{1}{v(t)}$ (general soln.)

$y' = \frac{-1}{t^2} - \frac{v'}{v^2} \Rightarrow \cancel{\frac{-1}{t^2}} - \frac{v'}{v^2} = \cancel{\frac{-1}{t^2}} - \frac{1}{t}\left(\frac{1}{t} + \frac{1}{v}\right) + \left(\frac{1}{t} + \frac{1}{v}\right)^2$

$\Rightarrow \frac{-v'}{v^2} = -\cancel{\frac{1}{t^2}} - \frac{1}{t}\cdot\frac{1}{v} + \cancel{\frac{1}{t^2}} + \frac{2}{t\cdot v} + \frac{1}{v^2} \Rightarrow -\frac{v'}{v^2} = \frac{1}{t}\cdot\frac{1}{v} + \frac{1}{v^2}$

$\begin{pmatrix}\text{multiplying by} \\ -v^2\end{pmatrix} \Rightarrow v' = -\frac{v}{t} - 1 \Rightarrow v' + \left(\frac{1}{t}\right)v = -1$ (linear eqn. in v)

$\Rightarrow \mu(t) = e^{\int \frac{1}{t}dt} = e^{\ln|t|} = |t|$

(Initial pt: $(1,-1)$) $\Rightarrow \mu(t) = t$

$\Rightarrow \frac{d}{dt}[v \cdot t] = -t \Rightarrow v \cdot t = -\int t\,dt = -\frac{t^2}{2} + c_1$

$\Rightarrow v(t) = \frac{1}{t}\left[c_1 - \frac{t^2}{2}\right] = \frac{2c_1 - t^2}{2t} \Rightarrow \frac{1}{v(t)} = \frac{2t}{2c_1 - t^2}$

$\Rightarrow y(t) = \frac{1}{t} + \frac{2t}{2c_1 - t^2} \xRightarrow[y(1)=-1]{} -1 = \frac{1}{1} + \frac{2}{2c_1 - 1} \Rightarrow \boxed{c_1 = 0}$

<u>Soln.:</u> $\boxed{y(t) = \frac{-1}{t}}$

3.) <u>I.V.P.:</u> $xy' + y - y^2 e^{2x} = 0 \quad , \quad y(1) = 2$

$$\Rightarrow y' + \frac{1}{x}y = \frac{e^{2x}}{x}y^2 \leftarrow \text{Bernoulli eqn. } (n=2)$$

$$\Rightarrow \underbrace{y^{-2}\cdot y' + \frac{1}{x}\,y^{-1}}_{} = \frac{e^{2x}}{x} \Rightarrow v = y^{-1} \Rightarrow \frac{dv}{dx} = -y^{-2}\cdot\frac{dy}{dx}$$

$$y^{-1} \downarrow v$$

$$\Rightarrow -\frac{dv}{dx} + \frac{1}{x}v = \frac{e^{2x}}{x} \Rightarrow v' - \frac{1}{x}v = -\frac{e^{2x}}{x} \leftarrow \text{linear eqn. in } v(x)$$

$$\mu(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln|x|} = |x|^{-1} = \frac{1}{|x|} = \frac{1}{x}$$

Initial pt: $(x_0, y_0) = (1, 2)$, $x_0 > 0$

$$\Rightarrow \frac{d}{dx}\left(v\cdot\frac{1}{x}\right) = \left(-\frac{e^{2x}}{x}\right)\left(\frac{1}{x}\right) = -x^{-2}\cdot e^{2x}$$

$$\Rightarrow v\cdot\frac{1}{x} = -\int \frac{e^{2x}}{x^2}dx \Rightarrow v(x) = -x\left[\int_1^x \frac{e^{2s}}{s^2}ds + C_1\right]$$

$$\Rightarrow \underbrace{v(x)}_{\frac{1}{y}} = -x\int_1^x \frac{e^{2s}}{s^2}ds + C_2 x$$

$$y(1) = 2 \Rightarrow \frac{1}{2} = -1\cancelto{0}{\int_1^1 \frac{e^{2s}}{s^2}ds} + C_2 \Rightarrow C_2 = \frac{1}{2}$$

<u>Soln.:</u> $$\boxed{\frac{1}{y} = -x\int_1^x \frac{e^{2s}}{s^2}ds + \frac{x}{2}}$$

4-) $\underbrace{((x+2)\sin y)}_{M}dx+\underbrace{(x\cos y)}_{N}dy=0 \Rightarrow \frac{\partial M}{\partial y}=(x+2)\cos y \neq \cos y = \frac{\partial N}{\partial x}$

$\Rightarrow$ Eqn. is not exact.

$$-\left(\frac{N_x-M_y}{N}\right)=-\left(\frac{\cos y-(x+2)\cos y}{x\cos y}\right)=-\frac{\cancel{\cos y}(1-x-2)}{x.\cancel{\cos y}}=\frac{x+1}{x}=\underbrace{1+\frac{1}{x}}_{\text{function of } x \text{ only}}$$

$$\Rightarrow \boxed{\underbrace{\mu(x)}_{\text{integrating factor}}}=e^{-\int\frac{N_x-M_y}{N}dx}=e^{\int\left(1+\frac{1}{x}\right)dx}=\boxed{xe^x}$$

$\Rightarrow xe^x((x+2)\sin y)dx+(xe^x)(x\cos y)dy=0 \longrightarrow$ exact eqn.

$\Rightarrow \underbrace{(x^2e^x+2xe^x)\sin y}_{\widetilde{M}}dx+\underbrace{(x^2e^x)\cos y}_{\widetilde{N}}dy=0 \Rightarrow \exists$ potential function $F(x,y) \ni$

$\frac{\partial F}{\partial x}=\widetilde{M}=(x^2e^x+2xe^x)\sin y$

$\frac{\partial F}{\partial y}=\widetilde{N}=x^2e^x\cos y$

and soln. is $F(x,y)=C$

$$\underbrace{\int\frac{\partial F}{\partial y}dy}_{F(x,y)}=\int x^2e^x\cos y\,dy=x^2e^x\sin y+h(x)$$

$$\left.\begin{array}{l}\frac{\partial F}{\partial x}=(2xe^x+x^2e^x)\sin y+h'(x)\\ \quad \| \\ (x^2e^x+2xe^x)\sin y\end{array}\right\} h'(x)=0 \Rightarrow h(x)=C_1$$

$\Rightarrow F(x,y)=x^2e^x\sin y+C_1=C$

$\Rightarrow$ <u>Soln.:</u> $\boxed{x^2e^x\sin y=D}$ $(D=C-C_1)$

5.) $\frac{dy}{d\theta} = \frac{\theta \sec(y/\theta) + y}{\theta} = \sec\left(\frac{y}{\theta}\right) + \frac{y}{\theta}$ ← homogeneous eqn.

$v = \frac{y}{\theta} \Rightarrow y = \theta . v \Rightarrow \frac{dy}{d\theta} = v + \theta . \frac{dv}{d\theta}$

$\Rightarrow \cancel{v} + \theta . \frac{dv}{d\theta} = \sec(v) + \cancel{v} \Rightarrow \frac{dv}{\sec(v)} = \frac{d\theta}{\theta}$ → separable eqn.

$\Rightarrow \int \cos(v)\, dv = \int \frac{d\theta}{\theta} \Rightarrow \sin(v) = \ln|\theta| + \ln c_1$

$\sin(v) = \ln(c_2 \theta) \quad (c_2 = \mp c_1)$

$\Rightarrow \underset{\frac{y}{\theta}}{\overset{\|}{v}} = \arcsin(\ln(c_2\theta))$

$\Rightarrow$ Soln.: $\boxed{y = \theta . \arcsin(\ln(c_2\theta))}$