Math 245 - Fall 2019

HW#2 Solutions:

(1-) $\boxed{y} = y_1 \cdot v(x) = \boxed{\frac{1}{x}\cdot v(x)} \Rightarrow y' = \frac{-v}{x^2} + \frac{v'}{x} \Rightarrow y'' = -\left(\frac{v'}{x^2} - \frac{2v}{x^3}\right) + \left(\frac{v''}{x} - \frac{v'}{x^2}\right)$

d.e.: $2x^2\left(\frac{v''}{x} - \frac{2v'}{x^2} + \frac{2v}{x^3}\right) + 3x\left(\frac{-v}{x^2} + \frac{v'}{x}\right) - \frac{v}{x} = 0$ (d.e.: $2x^2y'' + 3xy' - y = 0$)

$\Rightarrow 2xv'' - 4v' + \cancel{\frac{4v}{x}} - \cancel{\frac{3v}{x}} + 3v' - \cancel{\frac{v}{x}} = 0 \Rightarrow 2xv'' - v' = 0$

$\Rightarrow v'' - \frac{1}{2x}v' = 0$ Let $\left.\begin{matrix} w = v' \\ w' = v'' \end{matrix}\right\} \Rightarrow w' - \frac{1}{2x}w = 0 \Rightarrow \frac{dw}{w} = \frac{dx}{2x}$ (separable)

$\Rightarrow \ln|w| = \frac{1}{2}\ln|x| + \ln c_1 \Rightarrow \underset{\Downarrow\, v'}{w} = c_2\sqrt{x}$ $(c_2 = \sqrt{c_1})$

$\Rightarrow v = c_2\int\sqrt{x}\,dx = c_2\frac{x^{3/2}}{3/2} + c_3 = \frac{2}{3}c_2 x^{3/2} + c_3$

$\Rightarrow y = \frac{v(x)}{x} = \underbrace{\frac{2}{3}c_2}_{A} x^{1/2} + \underbrace{c_3}_{B} x^{-1} = A\underbrace{x^{1/2}}_{y_2} + \frac{B}{\underbrace{x}_{y_1}}$ } second lin. ind. soln. is: $\boxed{y_2 = Ax^{1/2}}$

(2-) $y'' + 2y' + 5y = 4e^{-t}\cos(2t),\ y(0) = 1,\ y'(0) = 0$:

y_h: $r^2 + 2r + 5 = 0 \Rightarrow (r+1)^2 + 4 = 0 \Rightarrow r + 1 = \mp 2i \Rightarrow r = -1 \mp 2i$

$\boxed{y_h(t) = c_1e^{-t}\cos(2t) + c_2e^{-t}\sin(2t)}$

$\boxed{y_p(t) = t\left[Ae^{-t}\cos(2t) + Be^{-t}\sin(2t)\right]}$

$y_p'(t) = \left[Ae^{-t}\cos(2t) + Be^{-t}\sin(2t)\right] + t\left[-Ae^{-t}\cos(2t) - Be^{-t}\sin(2t) - 2Ae^{-t}\sin(2t) + 2Be^{-t}\cos(2t)\right]$

$y_p'(t) = e^{-t}\left[(A + (2B - A)t)\cos(2t) + (B - (2A + B)t)\sin(2t)\right]$

$y_p''(t) = e^{-t}\left[((-3A - 4B)t - 2A + 4B)\cos(2t) + ((4A - 3B)t - 2B - 4A)\sin(2t)\right]$

into the d.e. $\Rightarrow A = 0,\ B = 1 \Rightarrow \boxed{y_p(t) = 1\cdot t\cdot e^{-t}\sin(2t)}$

$\Rightarrow y_g = y_h + y_p = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) + te^{-t}\sin(2t)$

I.C's: $\underbrace{y(0)}_{1} = c_1 \Rightarrow \boxed{c_1 = 1}$

$$y' = c_1\left[-e^{-t}\cos(2t) - 2e^{-t}\sin(2t)\right] + c_2\left[-e^{-t}\sin(2t) + 2e^{-t}\cos(2t)\right] + \left[e^{-t}\sin(2t) - te^{-t}\sin(2t) + 2te^{-t}\cos(2t)\right]$$

$c_1 = 1$

$\underbrace{y'(0)}_{0} = [-1] + c_2[2] + 0 \Rightarrow 2c_2 = 1 \Rightarrow \boxed{c_2 = \frac{1}{2}}$

$\Rightarrow$ Soln.: $\boxed{y_g = e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t) + te^{-t}\sin(2t)}$

(3.) Solve the Cauchy-Euler eqn.: $t^2y'' - 4ty' + 4y = \underbrace{3t^4}_{h(t)}$

$a = 1,\ b = -4,\ c = 4$

Substitution $t = e^x$ reduces the eqn. to: $ay''(x) + (b-a)y'(x) + cy = h(e^x)$

$\Rightarrow \boxed{y'' - 5y' + 4y = 3e^{4x}}$ (*) $\Rightarrow y_h(x)$: char. eqn.: $r^2 - 5r + 4 = 0$

$(r-4)(r-1) = 0$

$\Rightarrow \boxed{y_h(x) = c_1 e^{4x} + c_2 e^{x}}$

$$\left.\begin{aligned} \Rightarrow y_p(x) &= Axe^{4x} \\ y_p' &= A\left[e^{4x} + 4xe^{4x}\right] \\ y_p'' &= A\left[8e^{4x} + 16xe^{4x}\right] \end{aligned}\right\} \xRightarrow{\text{into } *} A = 1 \Rightarrow \boxed{y_p(x) = xe^{4x}}$$

$\Rightarrow y_g(x) = y_h(x) + y_p(x) = (c_1e^{4x} + c_2e^{x}) + (xe^{4x})$

$t = e^x \Leftrightarrow x = \ln t \Rightarrow y_g(t) = (c_1t^4 + c_2t) + (\ln t)(t^4)$

$\Rightarrow \boxed{y_g(t) = c_2t + (c_1 + \ln t)t^4}$

4 a) $y''+3y'-28y=7t+e^{-7t}+1$:

$$y_h(t):\ \left.\begin{array}{l} r^2+3r-28=0 \\ (r+7)(r-4)=0 \end{array}\right\} y_h(t)=c_1e^{-7t}+c_2e^{4t}$$

$$\Rightarrow \boxed{y_p(t)=(At+B)+Cte^{-7t}}$$

b.) $y''-100y=9t^2e^{10t}+\cos t-t\sin t$

$$y_h(t):\ \left.\begin{array}{l} r^2-100=0 \\ (r-10)(r+10)=0 \end{array}\right\} y_h(t)=c_1e^{10t}+c_2e^{-10t}$$

$$\Rightarrow \boxed{y_p(t)=t\left(At^2+Bt+C\right)e^{10t}+(Dt+E)\cos t+(Ft+G)\sin t}$$

c) $4y''+y=e^{-2t}\cdot\sin\left(\frac{t}{2}\right)+6t\cos\left(\frac{t}{2}\right)$

$$y_h(t):\ \left.\begin{array}{l} 4r^2+1=0 \\ r=\mp\frac{1}{2}i \\ \left(\alpha=0,\ \beta=\frac{1}{2}\right) \end{array}\right\} \begin{array}{l} y_h(t)=c_1\cos\left(\frac{t}{2}\right)+c_2\sin\left(\frac{t}{2}\right) \\ \boxed{y_p(t)=\left[Ae^{-2t}\sin\left(\frac{t}{2}\right)+Be^{-2t}\cos\left(\frac{t}{2}\right)\right]+\left[t(Ct+D)\cos\left(\frac{t}{2}\right)+t(Et+F)\sin\left(\frac{t}{2}\right)\right]} \end{array}$$

d.) $4y''+16y'+17y=e^{-2t}\cdot\sin\left(\frac{t}{2}\right)+6t\cdot\cos\left(\frac{t}{2}\right)$

$$y_h(t):\ \left.\begin{array}{l} 4r^2+16r+17=0 \\ (2r+4)^2+1=0 \\ r=-2\mp\frac{1}{2}i \\ \alpha=-2,\ \beta=\frac{1}{2} \end{array}\right\} \begin{array}{l} y_h(t)=c_1e^{-2t}\cos\left(\frac{t}{2}\right)+c_2e^{-2t}\sin\left(\frac{t}{2}\right) \\ \boxed{y_p(t)=t(At+B)e^{-2t}\cdot\cos\left(\frac{t}{2}\right)+t(Ct+D)e^{-2t}\cdot\sin\left(\frac{t}{2}\right)} \end{array}$$

e.) $y''+8y'+16y=e^{-4t}+(t^2+5)e^{-4t}$

$$y_h(t):\ \left.\begin{array}{l} r^2+8r+16=0 \\ (r+4)^2=0\Rightarrow r_1=r_2=-4 \end{array}\right\} y_h(t)=c_1e^{-4t}+c_2te^{-4t}$$

$$\Rightarrow \boxed{y_p(t)}=t^2\left(At^2+Bt+C\right)e^{-4t}=\boxed{\left(At^4+Bt^3+Ct^2\right)e^{-4t}}$$