Math 245-Fall 2019-HW#3 solutions

1-) $xy'' - (1+x)y' + y = x^2 e^{2x}, \ x>0$

a) $y_1 = x+1, \ y_2 = e^x$ solns. of $xy'' - (1+x)y' + y = 0$

$$W(y_1, y_2) = \begin{vmatrix} x+1 & e^x \\ 1 & e^x \end{vmatrix} = xe^x + e^x - e^x = xe^x \neq 0 \ (x>0)$$

Check also that y_1, y_2 satisfy the given hom. eqn.

b) V.P. method: $y_h = c_1 \underbrace{(x+1)}_{y_1} + c_2 \underbrace{(e^x)}_{y_2} \Rightarrow y_p = (v_1)(x+1) + (v_2)(e^x)$

$$v_1 = -\int \frac{\frac{\cancel{x^2} e^{2x}}{\cancel{x}} \cdot \cancel{e^x}}{\cancel{x e^x}} dx = -\int e^{2x} dx = \boxed{-\frac{e^{2x}}{2}}$$

$$v_2 = \int \frac{\frac{\cancel{x^2} e^{2x}}{\cancel{x}} \cdot (x+1)}{\cancel{x} \cdot e^x} dx = \int (x+1) e^x dx = (x+1)e^x - \int e^x dx = \boxed{xe^x}$$

$u = x+1 \Rightarrow du = dx$

$dv = e^x dx \Rightarrow v = e^x$

$$\Rightarrow y_p = \left(-\frac{e^{2x}}{2}\right)(x+1) + (xe^x)(e^x) = \frac{xe^{2x}}{2} - \frac{e^{2x}}{2} = \boxed{\frac{e^{2x}}{2}(x-1)}$$

$$\Rightarrow \boxed{y_g = y_h + y_p = c_1(x+1) + c_2(e^x) + \frac{e^{2x}}{2}(x-1)}$$

2-) $(3x+1)^2 y'' - 3(3x+1)y' + 9y = 18(3x+1), \ x > -\frac{1}{3}$

Let $3x+1 = e^t$

$3\frac{dx}{dt} = e^t$

$\Rightarrow \frac{dx}{dt} = \frac{1}{3}e^t$

$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \Rightarrow \boxed{\frac{dy}{dx} = 3e^{-t}\frac{dy}{dt}}$ (A)

$$\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)\left(\frac{dx}{dt}\right) = \frac{d^2y}{dx^2}\left(\frac{1}{9}e^{2t}\right) + \underbrace{\left(3e^{-t}\frac{dy}{dt} \cdot \frac{1}{3}e^t\right)}_{\frac{dy}{dt}}$$

$$\Rightarrow \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) = \frac{d^2y}{dx^2}\left(\frac{1}{9}e^{2t}\right)$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2} = 9e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)} \ \text{(B)}$$

Plugging Ⓐ & Ⓑ into the eqn., we get:

($\star$) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 2e^t$ (check this step!)

$y_h(t)$: $r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r_1 = r_2 = 1$

$\Rightarrow y_h(t) = c_1 e^t + c_2 t e^t$

$y_p(t) = At^2 e^t \Rightarrow y_p' = A(2te^t + t^2 e^t)$

$y_p'' = A(2e^t + 4te^t + t^2 e^t)$

Plugging into ($\star$) eqn. we get: $A = 1 \Rightarrow y_p(t) = t^2 e^t$

$\Rightarrow y_g(t) = c_1 e^t + c_2 t e^t + t^2 e^t$

$3x+1 = e^t \Rightarrow t = \ln(3x+1)$ (since $x > -\frac{1}{3}$ is given).

$\Rightarrow y_g(x) = c_1(3x+1) + c_2(3x+1)\ln(3x+1) + (3x+1)(\ln(3x+1))^2$

$$\boxed{y_g(x) = (3x+1)\left[c_1 + c_2 \ln(3x+1) + (\ln(3x+1))^2\right]}$$

3.) a) $\mathcal{L}\{t^2 \cdot e^{2t} \cdot \sin(3t)\}(s) = \mathcal{L}\{e^{2t} \cdot t^2 \sin(3t)\}(s)$

$= H(s-2) = \boxed{18\left[\frac{(s-2)^2-3}{((s-2)^2+9)^3}\right]}$

$\mathcal{L}\{t^2 \sin(3t)\}(s) = (-1)^2 \frac{d^2}{ds^2} \mathcal{L}\{\sin(3t)\}(s) = \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right) = \frac{d}{ds}\left(\frac{-6s}{(s^2+9)^2}\right)$

$= -\frac{6(s^2+9)^2 - 2(s^2+9)(2s)(6s)}{(s^2+9)^4} = -\frac{\cancel{(s^2+9)}\left[6(s^2+9) - 24s^2\right]}{(s^2+9)^{\cancel{4}\,3}}$

$= -6\left(\frac{9-3s^2}{(s^2+9)^3}\right) = 18\left(\frac{s^2-3}{(s^2+9)^3}\right) = H(s)$

b) $f(t) = \begin{cases} 0, & 0<t<2 \\ t, & t>2 \end{cases} \Rightarrow \mathcal{L}\{f(t)\}(s) = \int_0^2 0 \cdot e^{-st}\,dt + \int_2^\infty t \cdot e^{-st}\,dt$

$= \lim_{N\to\infty} \int_2^N t \cdot e^{-st}\,dt = \lim_{N\to\infty}\left[-\frac{te^{-st}}{s}\Big|_2^N + \frac{1}{s}\int_2^N e^{-st}\,dt\right]$

$u = t \Rightarrow du = dt$

$dv = e^{-st}dt \Rightarrow v = \frac{e^{-st}}{-s}$

$= \lim_{N\to\infty}\left[-\frac{N}{s}e^{-sN} + \frac{2}{s}e^{-2s} - \frac{1}{s^2}\left(e^{-sN} - e^{-2s}\right)\right] = \frac{2}{s}e^{-2s} + \frac{1}{s^2}e^{-2s}$

$= \boxed{e^{-2s}\left[\frac{2s+1}{s^2}\right]}$

c) $\mathcal{L}^{-1}\left\{\frac{2s^2+10s}{(s+1)(s^2-2s+5)}\right\}(t) = f(t)$

$\frac{2s^2+10s}{(s+1)(s^2-2s+5)} = \frac{A}{s+1} + \frac{Bs+C}{s^2-2s+5} = \frac{-1}{s+1} + \frac{3s+5}{(s-1)^2+(2)^2} = \frac{-1}{s+1} + 3\frac{s-1}{(s-1)^2+2^2} + 4 \cdot \frac{2}{(s-1)^2+2^2}$

$f(t) = \mathcal{L}^{-1}\left\{\frac{-1}{s+1}\right\}(t) + 3\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+2^2}\right\}(t) + 4\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\}(t)$

$= \boxed{-e^{-t} + 3e^{t} \cdot \cos(2t) + 4e^{t} \cdot \sin(2t)}$

d) $\mathcal{L}^{-1}\left\{\frac{8s^2-4s+12}{s(s^2+4)}\right\}(t)=f(t)$

$$\frac{8s^2-4s+12}{s(s^2+4)}=\frac{A}{s}+\frac{Bs+C}{s^2+4}=\frac{3}{s}+\frac{5s-4}{s^2+4}=\frac{3}{s}+5\frac{s}{s^2+4}-2\frac{2}{s^2+4}$$

$$f(t)=\mathcal{L}^{-1}\left\{\frac{3}{s}\right\}(t)+5\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}(t)-2\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}(t)$$

$$=\boxed{3+5\cos(2t)-2\sin(2t)}$$

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4) $y''+2y'+y=4e^{-t}$, $y(0)=2$, $y'(0)=-1$

$$\mathcal{L}\{y''(t)\}(s)+2\mathcal{L}\{y'(t)\}(s)+\mathcal{L}\{y(t)\}(s)=4\mathcal{L}\{e^{-t}\}(s)$$

$$\left[s^2Y(s)-s\underbrace{y(0)}_{2}-\underbrace{y'(0)}_{-1}\right]+2\left[sY(s)-\underbrace{y(0)}_{2}\right]+Y(s)=4\cdot\frac{1}{s+1}$$

$$Y(s)\underbrace{\left[s^2+2s+1\right]}_{(s+1)^2}=\frac{4}{s+1}+2s+3=\frac{2s^2+5s+7}{s+1}\Rightarrow Y(s)=\frac{2s^2+5s+7}{(s+1)^3}$$

$$\Rightarrow Y(s)=\frac{2}{s+1}+\frac{1}{(s+1)^2}+\frac{5}{(s+1)^3}$$

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}(t)=2e^{-t}+te^{-t}+\frac{5}{2}e^{-t}\cdot t^2$$

$$\boxed{y(t)=e^{-t}\left[2+t+\frac{5}{2}t^2\right]}$$

5) $y'(t)+y(t)-\underbrace{\int_0^t y(v)\cdot\sin(t-v)\,dv}_{y(t)*\sin t}=-\sin t,\quad y(0)=1$

$$\mathcal{L}\{y'(t)\}(s)+\mathcal{L}\{y(t)\}(s)-\mathcal{L}\{y(t)*\sin t\}(s)=-\mathcal{L}\{\sin t\}(s)$$

$$\left[sY(s)-\underbrace{y(0)}_{1}\right]+Y(s)-\left(Y(s)\cdot\frac{1}{s^2+1}\right)=-\frac{1}{s^2+1}$$

$$Y(s)\underbrace{\left[s+1-\frac{1}{s^2+1}\right]}_{\frac{s^3+s^2+s}{s^2+1}}=-\frac{1}{s^2+1}+1=\frac{s^2+1-1}{s^2+1}=\frac{s^2}{s^2+1}$$

$$\Rightarrow Y(s)=\frac{s^{\cancel{2}}}{\cancel{s^2+1}}\cdot\frac{\cancel{(s^2+1)}}{\cancel{s}(s^2+s+1)}=\frac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$=\frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}-\frac{1}{2}\cdot\frac{2}{\sqrt{3}}\cdot\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\boxed{y(t)=e^{-\frac{t}{2}}\cdot\cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{1}{\sqrt{3}}\,e^{-\frac{t}{2}}\cdot\sin\left(\frac{\sqrt{3}}{2}t\right)}$$