ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 245 - Differential Equations

2019-2020 Fall

FIRST MIDTERM EXAMINATION

11.11.2019, 17:30

SOLUTIONS

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR: B.K.

DURATION: 90 minutes

Question	Grade	Out of
1		20
2		20
3		20
4		20
5		20
Bonus		5
Total		105

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 6 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

1. (20 pts.) Determine whether the *Existence and Uniqueness Theorem for First order IVP's* implies that there is a unique solution for the below IVP. Why or why not? Show your work.

$$\underbrace{\frac{dy}{dx} = \sin x - \sqrt{y-2}}_{f(x,y)}, \quad \underbrace{y(\pi) = 2}_{\text{Initial pt.: } (x_0,y_0)=(\pi,2)}$$

$f(x,y) = \sin x - \sqrt{y-2}$

i) f is cont. near $(\pi, 2)$

ii) $\frac{\partial f}{\partial y} = -\frac{1}{2}(y-2)^{-1/2} = \frac{-1}{2\sqrt{y-2}}$ is not cont. at $y=1 \Rightarrow$

So the Existence & Uniqueness thm. does not guarantee a unique soln. of the I.V.P. near $(\pi, 2)$.

(Note that if the conditions of the theorem are not satisfied, this does not mean that there is NO unique solution. It simply means that the Existence and Uniqueness Theorem DOES NOT guarantee a unique solution.)

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2. (20 pts.) Obtain the general solution to the d.e.: $\frac{dy}{dx} = \frac{e^{x+y}}{y-1}$

$\frac{dy}{dx} = \frac{e^x . e^y}{y-1} \Rightarrow \int \frac{y-1}{e^y} dy = \int e^x dx$ (separable eqn.)

$\Rightarrow \int \underbrace{(y-1)}_{u} \underbrace{e^{-y} dy}_{dv} = e^x + C_1$

$u = y-1 \Rightarrow du = dy$

$dv = e^{-y} dy \Rightarrow v = -e^{-y}$

$\Rightarrow (y-1)(-e^{-y}) + \underbrace{\int e^{-y} dy}_{-e^{-y}} = e^x + C_1$

$-e^{-y}[y \cancel{-1} + \cancel{1}] = e^x + C_1$

$\frac{y}{e^y} = -e^x - C_1 \Rightarrow \boxed{y = -e^y [e^x + C_1]}$

3. (20 pts.) Solve the d.e.: $\underbrace{(\cos^2 t - y\cos t)}_{M(t,y)}\, dt - \underbrace{(1+\sin t)}_{N(t,y)}\, dy = 0$

$$\frac{\partial M}{\partial y} = -\cos t = \frac{\partial N}{\partial t} = -\cos t \Rightarrow \text{d.e. is } \underline{\text{exact}} \Rightarrow \exists \text{ potential function } F(t,y)$$

s.t. soln. is given by $F(t,y)=C$ and $\frac{\partial F}{\partial t} = M$, $\frac{\partial F}{\partial y} = N$.

$$\frac{\partial F}{\partial t} = M = \cos^2 t - y\cos t$$

$$\frac{\partial F}{\partial y} = N = -(1+\sin t) \Rightarrow$$

$$F(t,y) = -\int (1+\sin t)\,dy = -y(1+\sin t) + h(t)$$

$$\cos^2 t - y\cos t = \frac{\partial F}{\partial t} = -y\cos t + h'(t) \Rightarrow h'(t) = \cos^2 t$$

$$h(t) = \int \cos^2 t = \int \left(\frac{1+\cos 2t}{2}\right) dt$$

$$= \frac{1}{2}t + \frac{1}{2}\,\frac{\sin 2t}{2} + C_1$$

$$\Rightarrow F(t,y) = -y(1+\sin t) + \frac{t}{2} + \frac{\sin 2t}{4} + C_1 = C$$

Soln.: $\boxed{-y(1+\sin t) + \frac{t}{2} + \frac{\sin 2t}{4} = D}$

4. (20 pts.) Solve the I.V.P.: $\underbrace{(\tan y - 2)}_{M}\, dx + \underbrace{\left(x\sec^2 y + \frac{1}{y}\right)}_{N} dy = 0\,,\quad y(0) = 1.$

$$\frac{\partial M}{\partial y} = \sec^2 y = \frac{\partial N}{\partial x} = \sec^2 y \Rightarrow \text{equation is exact}$$

$\Rightarrow \exists$ a potential function $F(x,y) \ni$ soln. of the d.e. is given by $F(x,y) = C$ and $\begin{cases} \frac{\partial F}{\partial x} = M = (\tan y - 2) \\ \frac{\partial F}{\partial y} = N = \left(x \cdot \sec^2 y + \frac{1}{y}\right) \end{cases}$

$$\underbrace{\int \frac{\partial F}{\partial x}\, dx}_{F(x,y)} = \int (\tan y - 2)\, dx = x(\tan y - 2) + h(y)$$

$$\left.\begin{array}{l} \underbrace{\frac{\partial F}{\partial y}}_{x \cdot \sec^2 y + \frac{1}{y}} = x \cdot \sec^2 y + h'(y) \end{array}\right\} h'(y) = \frac{1}{y} \Rightarrow h(y) = \ln y + C_1$$

$$\Rightarrow F(x,y) = x(\tan y - 2) + \ln y + C_1 = C$$

$$\Rightarrow \boxed{x(\tan y - 2) + \ln y = D}$$

Applying the I.C.: $y(0) = 1 \Rightarrow 0(\underbrace{\cancel{\tan 1}}_{0} - 2) + \underbrace{\cancel{\ln 1}}_{0} = D \Rightarrow D = 0$

<u>Soln.:</u> $\boxed{x(\tan y - 2) + \ln y = 0}$

5. (20 pts.) Solve the I.V.P.: $\overbrace{(x^2+y^2)}^{M}\,dx + \overbrace{2xy}^{N}\,dy = 0\,,\quad y(1) = -1$
(Hint: use "*homogeneous*" and "*Bernoulli*" equation methods.)

Method I: $\dfrac{dy}{dx} = -\dfrac{x^2+y^2}{2xy} = -\dfrac{1}{2}\left[\left(\dfrac{x}{y}\right)+\left(\dfrac{y}{x}\right)\right] \rightarrow$ homogeneous eqn.

Let $v=\dfrac{y}{x} \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$v + x\frac{dv}{dx} = -\frac{1}{2}\left[\frac{1}{v}+v\right] = -\frac{1}{2v} - \frac{1}{2}v$$

$$x\frac{dv}{dx} = -\frac{1}{2v} - \frac{3}{2}v \Rightarrow \frac{dv}{dx} + \left(\frac{3}{2x}\right)v = \left(-\frac{1}{2x}\right)v^{-1} \rightarrow \text{Bernoulli eqn.}$$

$$v\frac{dv}{dx} + \left(\frac{3}{2x}\right)v^2 = -\frac{1}{2x}$$

Let $\boxed{v^2 = z}$ $\quad 2v\dfrac{dv}{dx} = \dfrac{dz}{dx}$

$$\Rightarrow \frac{1}{2}\frac{dz}{dx} + \left(\frac{3}{2x}\right)z = -\frac{1}{2x} \Rightarrow \frac{dz}{dx} + \left(\frac{3}{x}\right)z = -\frac{1}{x} \rightarrow \text{Linear eqn. in } z$$

$$\mu(x) = e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$$

$$\Rightarrow \frac{d}{dx}(x^3 \cdot z) = (x^3)\left(-\frac{1}{x}\right) = -x^2$$

$$\Rightarrow x^3 \cdot z = -\frac{x^3}{3} + C_1 \Rightarrow \underset{\overset{\parallel}{v^2}}{z} = -\frac{1}{3} + \frac{C_1}{x^3} \Rightarrow v^2 = \left(\frac{y}{x}\right)^2 = \frac{3C_1 - x^3}{3x^3}$$

$$\Rightarrow y^2 = x^2 \cdot \frac{3C_1 - x^3}{3x^3} = \frac{3C_1 - x^3}{3x}$$

$$\Rightarrow y = \mp\sqrt{\frac{3C_1 - x^3}{3x}}$$

I.C.: $y(1) = -1 \Rightarrow -1 = -\sqrt{\dfrac{3C_1 - 1}{3}} \Rightarrow 3C_1 - 1 = 3$

$3C_1 = 4$

$C_1 = \dfrac{4}{3}$ $\Rightarrow$ Soln.: $\boxed{y = -\sqrt{\dfrac{4 - x^3}{3x}}}$ $\left(3xy^2 + x^3 = 4\right)$

Method II: $\dfrac{\partial M}{\partial y} = 2y = \dfrac{\partial N}{\partial x} = 2y \Rightarrow$ eqn. is exact $\Rightarrow \exists F(x,y) \ni \begin{cases} \dfrac{\partial F}{\partial x} = x^2 + y^2 \\ \dfrac{\partial F}{\partial y} = 2xy \end{cases}$

$$\Rightarrow F(x,y) = \int 2xy\,dy = xy^2 + h(x)$$

$$\left.\begin{aligned} \underset{\overset{\parallel}{x^2+y^2}}{\frac{\partial F}{\partial x}} = y^2 + h'(x) \end{aligned}\right\} \Rightarrow h'(x) = x^2 \Rightarrow h(x) = \frac{x^3}{3} + C_1$$

$$\Rightarrow F(x,y) = xy^2 + \frac{x^3}{3} + C_1 = C_2$$

Soln.: $\boxed{xy^2 + \dfrac{x^3}{3} = C}$

$y(1) = -1 \Rightarrow 1(-1)^2 + \dfrac{1}{3} = C \Rightarrow C = \dfrac{4}{3}$

$\Rightarrow \boxed{3xy^2 + x^3 = 4}$

Bonus Question: (5 pts.) Solve the d.e.: $\frac{dy}{dx} = (\underbrace{x+y}_{z}+2)^2 = (z+2)^2$

$z = x+y \Rightarrow \frac{dz}{dx} = 1 + \frac{dy}{dx}$

$$\frac{dy}{dx} = \left(\frac{dz}{dx} - 1\right) = (z+2)^2 \Rightarrow \frac{dz}{dx} = (z+2)^2 + 1$$

$$\Rightarrow \underbrace{\int \frac{dz}{(z+2)^2+1}}_{\Downarrow} = \int dx = \boxed{x + C_1}$$

$u = z+2$

$du = dz$

$$\int \frac{du}{1+u^2} = \arctan(u) = \arctan(z+2)$$

$$= \boxed{\arctan(x+y+2) = x + C_1}$$

$$\Rightarrow x+y+2 = \tan(x+C_1)$$

$$\boxed{y = \tan(x+C_1) - x - 2}$$