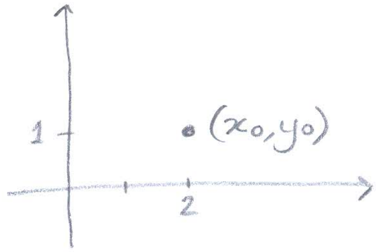
MCS 245-Differential Equations Fall 2012 Midterm #1

Instructor: B. Kaymakçalan Name: SOLUTIONS

Please show as much work as possible to get full or partial credit. Please circle your answers.

1. **(6 pts.)** Determine whether the *Existence and Uniqueness Theorem for First order IVP's* implies that there is a unique solution for the below IVP. Why or why not? Show your work.

$$\frac{dy}{dx} = \underbrace{3x - \sqrt[3]{y-1}}_{f(x,y)}, \quad y(2) = 1$$



Initial pt.: $(x_0, y_0) = (2,1)$

$f(x,y) = 3x - \sqrt[3]{y-1}$

i) f is cont. near $(2,1)$

ii) $\frac{\partial f}{\partial y} = \frac{-1}{3\sqrt[3]{(y-1)^2}}$ is not cont. at $y=1$ $\Rightarrow$ So the Existence & Uniqueness thm. does not guarantee a unique soln. of the I.V.P.

(Note that if the conditions of the theorem are not satisfied, this does not mean that there is NO unique solution. It simply means that the Existence and Uniqueness Theorem DOES NOT guarantee a unique solution.)

---

2. **(6 pts.)** Obtain the general solution to the d.e.: $\frac{dy}{dx} = x^2 e^{-4x} - 4y$

$\frac{dy}{dx} + 4y = x^2 \cdot e^{-4x}$ (linear eqn.)

Integrating factor: $\mu(x) = e^{\int 4dx} = e^{4x}$ $\Rightarrow$ multiplying both sides of eqn.

$\Rightarrow \frac{d}{dx}(e^{4x} \cdot y) = x^2 \cdot \cancel{e^{-4x}} \cdot \cancel{e^{4x}} = x^2 \Rightarrow e^{4x} \cdot y = \int x^2 dx = \frac{x^3}{3} + C_1$

$$\Rightarrow \boxed{y(x) = e^{-4x}\left(\frac{x^3}{3} + C_1\right)}$$

3. (18 pts.) Solve the d.e.: $(x^2+y^2)\,dx+2xy\,dy=0$

(Hint: use "*homogeneous*" and "*Bernoulli*" equation methods.)

$$\left.\begin{aligned}&\frac{dy}{dx}=-\frac{x^2+y^2}{2xy}=-\frac{x}{2y}-\frac{y}{2x}\\ &\text{Let } v=\frac{y}{x}\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}\end{aligned}\right\}\Rightarrow v+x\frac{dv}{dx}=-\frac{1}{2v}-\frac{v}{2}$$

$$\Rightarrow x\frac{dv}{dx}=-\frac{3}{2}v-\frac{1}{2}v^{-1}\Rightarrow\frac{dv}{dx}+\frac{3}{2x}v=-\frac{1}{2x}v^{-1}$$

$n=-1$ (n^{th} order Bernoulli eqn.)

$$z=v^{1-n}=v^{1-(-1)}=v^2\Rightarrow\frac{dz}{dx}=2v\frac{dv}{dx}$$

$$\Rightarrow 2v\frac{dv}{dx}+\frac{3}{x}v^2=-\frac{1}{x}\Rightarrow\frac{dz}{dx}+\frac{3}{x}z=-\frac{1}{x}\quad\text{(linear eqn.)}$$

$$\Rightarrow\mu(x)=e^{\int\frac{3}{x}dx}=e^{3\ln x}=x^3\quad(x>0)$$

$$\Rightarrow\frac{d}{dx}(z.x^3)=-\frac{1}{x}.x^3=-x^2\Rightarrow z.x^3=-\frac{x^3}{3}+C_1$$

$$\boxed{z(x)=-\frac{1}{3}+\frac{C_1}{x^3}}$$

$$\Rightarrow z=v^2=\left(\frac{y}{x}\right)^2=-\frac{1}{3}+\frac{C_1}{x^3}\Rightarrow\boxed{y^2=-\frac{1}{3}x^2+\frac{C_1}{x}}\quad\text{implicit soln.}$$

$$\Rightarrow\boxed{y(x)=\mp\sqrt{-\frac{1}{3}x^2+\frac{C_1}{x}}}$$

4. (18 pts.) Solve the IVP: $\underbrace{(\tan y-2)}_{M}\,dx+\underbrace{(x\sec^2 y+\frac{1}{y})}_{N}\,dy=0,\quad y(0)=1$

$\frac{\partial M}{\partial y}=\sec^2 y=\frac{\partial N}{\partial x}=\sec^2 y \Rightarrow$ eqn. is exact $\Rightarrow$

Soln. is of the form: $F(x,y)=C$ where

$\frac{\partial F}{\partial x}=\tan y-2 \Rightarrow F(x,y)=(\tan y-2)x+h(y)$

$\frac{\partial F}{\partial y}=\underbrace{x\sec^2 y+\frac{1}{y}}$

$\frac{\partial F}{\partial y}=(\sec^2 y).x+h'(y)\overset{\vee}{=}x.\sec^2 y+\frac{1}{y} \Rightarrow h'(y)=\frac{1}{y}$

$\Rightarrow h(y)=\ln|y|+C_1$

$\Rightarrow F(x,y)=x\cdot(\tan y-2)+\ln|y|+C_1=C$

Soln.: $\boxed{x(\tan y-2)+\ln|y|=D}$ $(D=C-C_1)$

5. Find the form of the **particular solution** for the method of undetermined coefficients.
DO NOT EVALUATE THE COEFFICIENTS.

(9 pts.) (a) $y''+2y'+10y=8t^4e^{-t}\sin 3t$,

char. eqn.: $r^2+2r+10=0 \Rightarrow (r+1)^2+9=0 \Rightarrow r_{1,2}=-1\mp 3i$

$y_c = c_1e^{-t}\cdot\cos 3t + c_2e^{-t}\sin 3t$

$$\boxed{\begin{aligned} y_p = {} & t(A_1t^4+B_1t^3+C_1t^2+D_1t+E_1)e^{-t}\sin 3t \\ & + t(A_2t^4+B_2t^3+C_2t^2+D_2t+E_2)e^{-t}\cos 3t \end{aligned}}$$

(9 pts.) (b) $y''-y'-12y=2t^5e^{4t}$

char. eqn.: $r^2-r-12=0 \Rightarrow (r-4)(r+3)=0$

$y_c = c_1e^{4t}+c_2e^{-3t}$

$$\boxed{y_p = t(At^5+Bt^4+Ct^3+Dt^2+Et+F)e^{4t}}$$

6. **(12 pts.)** Find the general solution to the IVP using *method of undetermined coefficients* (UC method)

$$y''+2y'+y=t^2+1-e^{-t},\quad y(0)=0,\quad y'(0)=0.$$

char. eqn.: $r^2+2r+1=0 \Rightarrow (r+1)^2=0$

$y_c=c_1e^{-t}+c_2te^{-t}$

$y_p=(At^2+Bt+C)+t^2(De^{-t})$

$y_p'=(2At+B)+D(2te^{-t}-t^2e^{-t})$

$y_p''=(2A)+D(2e^{-t}-2te^{-t}-2te^{-t}+t^2e^{-t})$

into d.e.: $2A+D(\cancel{t^2e^{-t}}-\cancel{4te^{-t}}+2e^{-t})+2(2At+B+\cancel{2Dte^{-t}}-\cancel{Dt^2e^{-t}})$
$+(At^2+Bt+C)+t^2(\cancel{De^{-t}})=t^2+1-e^{-t}$

e^{-t}: $2D=-1 \Rightarrow \boxed{D=-\frac{1}{2}}$

t^2: $A=1 \Rightarrow \boxed{A=1}$

t: $4A+B=0 \Rightarrow \boxed{B=-4}$

const.: $2A+2B+C=1 \Rightarrow \boxed{C=7}$

$y_p=(t^2-4t+7)-\frac{1}{2}t^2e^{-t} \Rightarrow \boxed{y_g=c_1e^{-t}+c_2te^{-t}+(t^2-4t+7)-\frac{1}{2}t^2e^{-t}}$

$y_g(0)=c_1+7=0 \Rightarrow \boxed{c_1=-7}$

$y_g'(t)=+7e^{-t}+c_2(e^{-t}-te^{-t})+(2t-4)-\frac{1}{2}(2te^{-t}-t^2e^{-t})$

$y_g'(0)=7+c_2-4=0 \Rightarrow \boxed{c_2=-3}$

$$\boxed{y_g(t)=-7e^{-t}-3te^{-t}+(t^2-4t+7)-\frac{1}{2}t^2e^{-t}}$$

7. (18 pts.) a) Find a <u>general solution</u> to the equation $y''-6y'+9y=x^{-3}e^{3x}$, $x>0$. (use VP method)

char. eqn.: $r^2-6r+9=0 \Rightarrow (r-3)^2=0 \Rightarrow y_c(x)=c_1\underbrace{e^{3x}}_{y_1}+c_2\underbrace{xe^{3x}}_{y_2}$

$$W(y_1,y_2)(x)=\begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x}+3xe^{3x} \end{vmatrix} = e^{6x}+\cancel{3xe^{6x}}-\cancel{3xe^{6x}}=e^{6x}\neq 0$$

$$y_p(x)=v_1(x).e^{3x}+v_2(x).(xe^{3x})$$

$$v_1'(x)=\frac{\begin{vmatrix} 0 & xe^{3x} \\ x^{-3}e^{3x} & e^{3x}+3xe^{3x} \end{vmatrix}}{e^{6x}} = -\frac{x^{-2}\cdot\cancel{e^{6x}}}{\cancel{e^{6x}}} \Rightarrow \boxed{v_1(x)=x^{-1}}$$

$$v_2'(x)=\frac{\begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & x^{-3}\cdot e^{3x} \end{vmatrix}}{e^{6x}} = \frac{x^{-3}\cdot e^{6x}}{e^{6x}} = x^{-3} \Rightarrow \boxed{v_2(x)=-\frac{x^{-2}}{2}}$$

$$y_p(x)=(x^{-1})(e^{3x})+\left(-\frac{x^{-2}}{2}\right)(xe^{3x})=\boxed{\frac{e^{3x}}{2x}}$$

$$y_g(x)=y_c(x)+y_p(x)=\boxed{c_1e^{3x}+c_2xe^{3x}+\frac{e^{3x}}{2x}}$$

(4 pts.) b) Explicitly identify the complementary solution, $y_c(x)$, as well as the particular solution, $y_p(x)$.

$$\boxed{\begin{aligned} y_c(x)&=c_1e^{3x}+c_2xe^{3x} \\ y_p(x)&=\frac{e^{3x}}{2x} \end{aligned}}$$

Extra Credit Problem: (10 points)

Show that the equation $(\underbrace{2y^2-6xy}_{M})\,dx+(\underbrace{3xy-4x^2}_{N})\,dy=0$ is not exact. Find an integrating factor of the form $\mu(x,y)=x^n y^m$ that would make this equation exact and verify that the new equation is exact. DO NOT solve the equation.

$$\frac{\partial M}{\partial y}=4y-6x \neq \frac{\partial N}{\partial x}=3y-8x \Rightarrow \text{given d.e. is not exact}$$

$$\underbrace{\left[x^n y^m(2y^2-6xy)\right]}_{M_1}dx+\underbrace{\left[x^n y^m(3xy-4x^2)\right]}_{N_1}dy=0 \quad (A)$$

$$\left.\begin{aligned} M_1&=2x^n y^{m+2}-6x^{n+1}\cdot y^{m+1}\\ N_1&=3x^{n+1}\cdot y^{m+1}-4x^{n+2}\cdot y^m\end{aligned}\right\}\quad \begin{aligned}\frac{\partial M_1}{\partial y}&=2(m+2)x^n\cdot y^{m+1}-6(m+1)x^{n+1}\cdot y^m\\ \frac{\partial N_1}{\partial x}&=3(n+1)x^n\cdot y^{m+1}-4(n+2)x^{n+1}\cdot y^m\end{aligned}$$

$$\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}\Rightarrow \left.\begin{aligned}2(m+2)=3(n+1)&\Rightarrow 2m-3n=-1\\ 6(m+1)=4(n+2)&\Rightarrow 6m-4n=2\end{aligned}\right\}\Rightarrow \boxed{\begin{aligned}n&=1\\ m&=1\end{aligned}}$$

$\boxed{\mu(x,y)=x\cdot y}$ makes the eqn. exact!

Check that (A) is exact:

$$\frac{\partial M_1}{\partial y}=6xy^2-12x^2y = \frac{\partial N_1}{\partial x}=6xy^2-12x^2y$$

So new eqn. (A) is exact!