MCS 245-Differential Equations Fall 2011 Midterm #1

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Please show as much work as possible to get full or partial credit. Please circle your answers.

1. **(8 p)** For the equation $8\frac{d^4y}{dx^4} = x(1-x)$ answer the following

a) Is this ordinary or partial differential equation (DE)? Ordinary (ODE)

b) What is the order of this DE? 4 (four)

c) What is the dependent variable and what is the independent variable? $y \to$ dependent, $x \to$ independent

d) Is this a linear or nonlinear equation? $\boxed{\text{LINEAR}}$

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2. **(9 p)** Classify the following equations as separable, linear, exact, homogeneous (v=y/x type) or none of them. Note that some equations may have more than one classification. Justify.

a) $\underbrace{y^2}_{M}dx + \underbrace{(2xy+\cos y)}_{N}dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{y^2}{2xy+\cos y}$ } not seperable, not linear, not homogeneous

$\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x} = 2y \Rightarrow \boxed{\text{exact}}$ ALSO: $\frac{dx}{dy} + x\cdot\frac{y}{2} = -\frac{\cos y}{y^2}$ } $\boxed{\text{LINEAR in } x}$

b) $yxdx + dy = 0 \Rightarrow \frac{dy}{dx} = -yx \to$ not exact, not homogeneous

$\frac{dy}{dx} + x\cdot y = 0 \Rightarrow \boxed{\text{linear}}$ ALSO $\frac{dy}{y} = -xdx \to \boxed{\text{separable}}$

c) $(ye^{-2x} + y^3)dx - e^{-2x}dy = 0$

$\frac{dy}{dx} = \frac{ye^{-2x}+y^3}{e^{-2x}} = y + e^{2x}\cdot y^3 \Rightarrow \frac{dy}{dx} - y = e^{2x}\cdot y^3 \to$ Bernoulli eqn.

$v = y^{1-3} = y^{-2}$ transforms d.e. to: $\boxed{\frac{dv}{dx} + 2v = -2e^{2x}}$ hence $\boxed{\text{LINEAR}}$

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3. **(5 p)** Determine if the relation $x^2 - \sin(x+y) = 1$ is the implicit solution to the equation $\frac{dy}{dx} = 2x\sec(x+y) - 1$.

$2x - \cos(x+y)\cdot\left[1+\frac{dy}{dx}\right] = 0 \Rightarrow 2x - \cos(x+y) = \cos(x+y)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos(x+y)}\cdot\left[2x - \cos(x+y)\right] = \boxed{2x\cdot\sec(x+y) - 1}$

$\frac{1}{\cos(x+y)} = \sec(x+y)$

$\Rightarrow$ Yes, $x^2 - \sin(x+y) = 1$ satisfies the d.e. $\Rightarrow$ hence it is an implicit soln.

4. (12 pts.) Find the solution to the equation $4y''-4y'+5y=0,\ y(0)=1, y'(0)=-11/2$.

char. eqn.: $4r^2-4r+5=0 \Rightarrow r_{1,2}=\dfrac{4\mp\sqrt{16-80}}{8}=\dfrac{1}{2}\mp i$

$y(t)=c_1 e^{\frac{t}{2}}\cos t+c_2 e^{\frac{t}{2}}\sin t$

$y(0)=c_1=1 \Rightarrow \boxed{c_1=1}$

$y'(t)=\frac{1}{2}e^{\frac{t}{2}}\cdot\cos t+e^{\frac{t}{2}}(-\sin t)+c_2\frac{1}{2}e^{\frac{t}{2}}\sin t+c_2 e^{\frac{t}{2}}(\cos t)$

$y'(0)=\frac{1}{2}+c_2=-\frac{11}{2} \Rightarrow \boxed{c_2}=-\frac{11}{2}-\frac{1}{2}=\boxed{-6}$

$$\boxed{y(t)=e^{\frac{t}{2}}\cdot\cos t-6e^{\frac{t}{2}}\cdot\sin t}$$

5. Find the form of the particular solution for the method of undetermined coefficients. **DO NOT EVALUATE THE COEFFICIENTS.**

(9 pts.) (a) $y''+2y'+2y = 8t^3e^{-t}\sin t$,

$y_c \Rightarrow \underbrace{r^2+2r+2=0}_{\text{char. eqn.}} \Rightarrow (r+1)^2+1=0 \Rightarrow r=-1\mp i$

$y_c(t) = c_1e^{-t}\cos t + c_2e^{-t}\sin t$

$y_p(t) = t\,(A_1t^3+B_1t^2+C_1t+D_1)\,e^{-t}\cos t + t(A_2t^3+B_2t^2+C_2t+D_2)e^{-t}\sin t$

$= \boxed{(A_1t^4+B_1t^3+C_1t^2+D_1t)e^{-t}\cos t + (A_2t^4+B_2t^3+C_2t^2+D_2t)e^{-t}\sin t}$

(9 pts.) b) $y''-y'-12y = 2t^6e^{-3t}$

char. eqn.: $r^2-r-12=0 \Rightarrow (r-4)(r+3)=0 \Rightarrow r_1=4,\ r_2=-3$

$y_c(t) = c_1e^{4t} + c_2e^{-3t}$

$\boxed{y_p(t) = t\,(At^6+Bt^5+Ct^4+Dt^3+Et^2+Ft+G)\,e^{-3t}}$

6.(22 pts.) Find the solution to the IVP using method of undetermined coefficients (UC method)

$$10y''+y'-3y=t-e^{\frac{t}{2}},\ y(0)=\frac{8}{9},\ y'(0)=\frac{19}{33}.$$

char. eqn. $\Rightarrow 10r^2+r-3=0 \Rightarrow r_{1,2}=\dfrac{-1\mp\sqrt{1+120}}{20}=\dfrac{-1\mp 11}{20}$

$$\boxed{r_1=-\frac{3}{5} \quad r_2=\frac{1}{2}}$$

$$\boxed{y_c(t)=c_1e^{-\frac{3}{5}t}+c_2e^{\frac{1}{2}t}}$$

$$\boxed{y_p(t)=(At+B)+tCe^{\frac{t}{2}}}$$

$$y_p'(t)=A+C\left(e^{\frac{t}{2}}+\frac{t}{2}e^{\frac{t}{2}}\right)=A+e^{\frac{t}{2}}\cdot C\left(1+\frac{t}{2}\right)$$

$$y_p''(t)=C\left(\frac{1}{2}e^{\frac{t}{2}}+\frac{1}{2}\left[e^{\frac{t}{2}}+\frac{t}{2}e^{\frac{t}{2}}\right]\right)=e^{\frac{t}{2}}\cdot C\left[1+\frac{t}{4}\right]$$

D.E. $\Rightarrow e^{\frac{t}{2}}\left[10C\left(1+\frac{t}{4}\right)+C\left(1+\frac{t}{2}\right)-3Ct\right]+A-3(At+B)=t-e^{\frac{t}{2}}$

$$\begin{cases} 11C+Ct\underbrace{\left(\cancel{\frac{5}{2}+\frac{1}{2}-3}\right)}_{0}=-1 \Rightarrow 11C=-1 \Rightarrow \boxed{C=-\frac{1}{11}} \\ -3At+(A-3B)=t \Rightarrow \begin{array}{l} -3A=1 \Rightarrow \boxed{A=-\frac{1}{3}} \\ A-3B=0 \Rightarrow \boxed{B=-\frac{1}{9}} \end{array} \end{cases}$$

$$y(t)=y_c(t)+y_p(t)=c_1e^{-\frac{3}{5}t}+c_2e^{\frac{t}{2}}-\frac{1}{3}\left(t+\frac{1}{3}\right)-\frac{t}{11}e^{\frac{t}{2}}$$

$$y(0)=c_1+c_2-\frac{1}{9}=\frac{8}{9} \Rightarrow \boxed{c_1+c_2=1}$$

$$y'(0)=c_1\left(-\frac{3}{5}\right)+c_2\left(\frac{1}{2}\right)-\frac{1}{3}-\frac{1}{11}(1+0)=\frac{19}{33} \Rightarrow \boxed{-6c_1+5c_2=10}$$

$$\Rightarrow c_2=\frac{16}{11},\ c_1=-\frac{5}{11}$$

$$\boxed{y(t)=-\frac{5}{11}e^{-\frac{t}{5}}+\frac{16}{11}e^{\frac{t}{2}}-\frac{t}{3}-\frac{1}{9}-\frac{te^{\frac{t}{2}}}{11}}$$

7.(22 pts.) a) Find a <u>general solution</u> to the equation $y''-6y'+9y=x^{-3}e^{3x}$, $x>0$. (use VP method)

$r^2-6r+9=0 \Rightarrow (r-3)^2=0 \Rightarrow y_c(x)=c_1e^{3x}+c_2xe^{3x}$

$y_p(x)=v_1(x)\cdot\underbrace{e^{3x}}_{y_1(x)}+v_2(x)\cdot\underbrace{xe^{3x}}_{y_2(x)}$

$$W(y_1,y_2)(x)=\begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x}+3xe^{3x}\end{vmatrix} = e^{6x}\neq 0$$

$$v_1(x)=\int \frac{-\frac{x^{-3}\cdot e^{3x}}{1}\cdot xe^{3x}}{e^{6x}}\,dx$$

$$=-\int x^{-2}dx=-\frac{x^{-1}}{-1}=x^{-1}=\boxed{\frac{1}{x}}$$

$$v_2(x)=\int\frac{\frac{x^{-3}\cdot e^{3x}}{1}\cdot e^{3x}}{e^{6x}}dx=\int x^{-3}dx=\frac{x^{-2}}{-2}=\boxed{\frac{-1}{2x^2}}$$

$$\Rightarrow y_p(x)=\left(\frac{1}{x}\right)e^{3x}+\left(\frac{-1}{2x^2}\right)\left(xe^{3x}\right)=\frac{1}{x}e^{3x}-\frac{1}{2x}e^{3x}=\boxed{\frac{e^{3x}}{2x}}$$

$$\boxed{y_g(x)=c_1e^{3x}+c_2xe^{3x}+\frac{e^{3x}}{2x}}$$

(4 pts.) b) Explicitly identify the complementary solution, $y_c(x)$, as well as the particular solution, $Y_p(x)$.

$$y_g(x)=\underbrace{c_1e^{3x}+c_2xe^{3x}}_{y_c(x)}+\underbrace{\frac{e^{3x}}{2x}}_{y_p(x)}$$

Extra Credit Problem: (10 points)

Show that the equation $\underbrace{(2x^2+y)}_{M}dx+\underbrace{(x^2y-x)}_{N}dy=0$ is not exact. Find the integrating factor that would make this equation exact and verify that the new equation is exact. DO NOT solve the equation.

$$\frac{\partial M}{\partial y}=1\neq\frac{\partial N}{\partial x}=2xy-1 \Rightarrow \text{not exact}$$

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{1-(2xy-1)}{x^2y-x}=\frac{2-2xy}{x(xy-1)}=\frac{2\cancel{(1-xy)}}{-x\cancel{(1-xy)}}=\boxed{-\frac{2}{x}}$$

func. of x only

$$\mu(x)=e^{\int -\frac{2}{x}dx}=e^{-2\ln x}=x^{-2}=\boxed{\frac{1}{x^2}} \rightarrow \text{integrating factor}$$

$$\left(\frac{1}{x^2}\right)(2x^2+y)dx+\left(\frac{1}{x^2}\right)(x^2y-x)dy=0$$

$$\boxed{\underbrace{\left(2+\frac{y}{x^2}\right)}_{M_1}dx+\underbrace{\left(y-\frac{1}{x}\right)}_{N_1}dy=0} \quad (\star)$$

$$\left.\begin{aligned}\frac{\partial M_1}{\partial y}&=\frac{1}{x^2}\\ \frac{\partial N_1}{\partial x}&=\frac{1}{x^2}\end{aligned}\right\}$$ since these are equal the new equation $(\star)$ is exact!