ÇANKAYA UNIVERSITY
DEPARTMENT OF MATHEMATICS

# Math 245 Differential Equations

**2018 - 2019 FALL**

## FINAL EXAM

**January 9, 2019, 12:30-14:10**

-SOLUTIONS-

**NAME, SURNAME** :

**NUMBER** :

**DEPARTMENT** :

**SIGNATURE** :

**INSTRUCTOR'S NAME** : **Prof. Dr. Billur Kaymakçalan**

**DURATION** : **100 minutes**

**TOTAL NUMBER OF QUESTIONS** : **5+2**

| Questions | Grade | Out of |
|---|---|---|
| 1 | | 16 |
| 2 | | 20 |
| 3 | | 20 |
| 4 | | 20 |
| 5 | | 24 |
| EC I | | 10 |
| EC II | | 10 |
| TOTAL | | 120 |

**IMPORTANT NOTES:**

1) Please make sure that you have written your student number and name above.
2) Check that the exam paper contains 7 questions and 6 pages.
3) Read the questions carefully and show all your work. No points will be given to correct answers without reasonable work.
4) Usage of any kind of electronical equipments is not allowed.

(16 pts.) 1-) Solve the D.E.: $\underbrace{(\tan y-2)}_{M}dx+\underbrace{(x\sec^2 y+\frac{1}{y})}_{N}dy=0$

$\frac{\partial M}{\partial y}=\sec^2 y=\frac{\partial N}{\partial x} \Rightarrow$ D.E. is exact $\Rightarrow \exists$ potential function

$F(x,y) \ni \frac{\partial F}{\partial x}=M=\tan y-2$, $\frac{\partial F}{\partial y}=N=x\sec^2 y+\frac{1}{y}$ and the

soln. of the D.E. is given by $F(x,y)=C$.

$\frac{\partial F}{\partial x}=\tan y-2 \Rightarrow F(x,y)=x\tan y-2x+g(y)$

$x\sec^2 y+\frac{1}{y}=\frac{\partial F}{\partial y}=x\sec^2 y+g'(y) \Rightarrow g'(y)=\frac{1}{y} \Rightarrow g(y)=\ln y+C_1$

$\Rightarrow$ Soln.: $\boxed{x\tan y-2x+\ln y=C}$ //

(20 pts.) 2) Solve the I.V.P.: $y''-4y'+4y=7te^{2t}$

$y(0)=1\,,\ y'(0)=0$

char. eqn.: $r^2-4r+4=0 \Rightarrow (r-2)^2=0 \Rightarrow r_1=r_2=2$

$y_c(t)=c_1e^{2t}+c_2te^{2t}$

$$\left.\begin{aligned} y_p(t)&=t^2(At+B)e^{2t}=(At^3+Bt^2)e^{2t}\\ y_p'(t)&=e^{2t}(2At^3+t^2(2B+3A)+t(2B))\\ y_p''(t)&=e^{2t}(4At^3+t^2(12A+4B)+t(8B+6A)+2B)\end{aligned}\right\}\ \text{into the d.e.} \Rightarrow A=\tfrac{1}{2},\ B=0$$

$\Rightarrow y_p(t)=\frac{1}{2}t^3e^{2t}$

$\Rightarrow y_g(t)=y_c(t)+y_p(t)=c_1e^{2t}+c_2te^{2t}+\frac{1}{2}t^3e^{2t}$

$y(0)=c_1=1 \Rightarrow \boxed{c_1=1}$

$y'(t)=2c_1e^{2t}+c_2\left[e^{2t}+2te^{2t}\right]+\frac{1}{2}\left[3t^2e^{2t}+2t^3e^{2t}\right]$

$y'(0)=2(1)+c_2=0 \Rightarrow \boxed{c_2=-2}$

$\Rightarrow \boxed{y_g(t)=e^{2t}-2te^{2t}+\frac{1}{2}t^3e^{2t}}$ //

(20 pts.) 3.) Solve the D.E.: $y''+y=\tan^2 x$

$y_c(x)$: char. eqn.; $r^2+1=0 \Rightarrow r=\mp i \Rightarrow y_c(x)=c_1\underbrace{\cos x}_{y_1}+c_2\underbrace{\sin x}_{y_2}$

$$W(y_1,y_2)(x)=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}=\cos^2 x+\sin^2 x=1$$

Using V.P. method; $\boxed{y_p(x)=v_1(x)\cdot\cos x+v_2(x)\cdot\sin x}$

$$v_1(x)=\int\frac{-\tan^2 x.\sin x}{1}dx=-\int\frac{\overbrace{\sin^2 x}^{(1-\cos^2 x)}}{\cos^2 x}.\sin x\,dx=\int\left(\frac{1}{\cos^2 x}-1\right)(-\sin x)dx$$

$u=\cos x$
$du=-\sin x\,dx$

$$=\int\left(\frac{1}{u^2}-1\right)du$$
$$=-\frac{1}{u}-u$$
$$=\boxed{-\sec x-\cos x}$$

$$\boxed{v_1(x)=-\sec x-\cos x}$$

$$\boxed{v_2(x)}=\int\frac{\tan^2 x.\cos x}{1}dx=\int\frac{\sin^2 x}{\cos x}dx=\int\frac{1-\cos^2 x}{\cos x}dx$$
$$=\int(\sec x-\cos x)dx=\boxed{\ln|\sec x+\tan x|-\sin x}$$

$$\Rightarrow\boxed{y_p(x)}=(-\sec x-\cos x)(\cos x)+(\ln|\sec x+\tan x|-\sin x)(\sin x)$$
$$=-1-\cos^2 x+(\ln|\sec x+\tan x|)(\sin x)-\sin^2 x$$
$$=\boxed{(\sin x)(\ln|\sec x+\tan x|)-2}$$

$$\Rightarrow y_g(x)=y_c(x)+y_p(x)=\underbrace{c_1\cos x+c_2\sin x}_{y_c(x)}+\underbrace{(\sin x)(\ln|\sec x+\tan x|)-2}_{y_p(x)}$$

(20pts.) 4.) Solve the following I.V.P. <u>using Laplace transform</u>:

$$ty''+y'=t \;,\; y(0)=0 \;,\; y'(0)=0$$

$$\mathcal{L}\{ty''(t)\}(s)+\mathcal{L}\{y'(t)\}(s)=\mathcal{L}\{t\}(s)$$

$$-\frac{d}{ds}\Big[s^2Y(s)-s\underbrace{\cancel{y(0)}}_{0}-\underbrace{\cancel{y'(0)}}_{0}\Big]+\Big[sY(s)-\underbrace{\cancel{y(0)}}_{0}\Big]=\frac{1}{s^2}$$

$$-\big[2s.Y(s)+s^2Y'(s)\big]+sY(s)=\frac{1}{s^2}\Rightarrow s^2Y'(s)+sY(s)=-\frac{1}{s^2}$$

$$\Rightarrow \underbrace{Y'(s)+\frac{1}{s}Y(s)=-\frac{1}{s^4}}_{\text{linear eqn. in } Y(s)} \Rightarrow \mu(s)=e^{\int\frac{1}{s}ds}=e^{\ln s}=s$$

$$\Rightarrow \frac{d}{ds}\big[Y(s).\,s\big]=(s)\left(-\frac{1}{s^4}\right)=-\frac{1}{s^3}\Rightarrow Y(s)=\frac{1}{s}\left(\int-\frac{1}{s^3}ds\right)$$

$$\Rightarrow Y(s)=\frac{1}{s}\left[\frac{1}{2s^2}+c_1\right]=\frac{1}{2s^3}+\frac{c_1}{s}$$

$$\Rightarrow \boxed{y(t)}=\mathcal{L}^{-1}\{Y(s)\}(t)=\boxed{\frac{1}{4}t^2+c_1}$$

$$y(0)=c_1=0\Rightarrow \boxed{c_1=0}\;;\; y'(t)=\frac{t}{2}\;,\; y'(0)=0$$

<u>So</u>; <u>soln.</u>: $\boxed{y(t)=\dfrac{t^2}{4}}$

5-) Find the Inverse Laplace transform of the followings:

(8pts.) a.) $F(s)=\dfrac{s-1}{2s^2+s+6}=\dfrac{\left(s+\frac{1}{2}\right)-\frac{3}{2}}{2\left[\left(s+\frac{1}{2}\right)^2+\frac{13}{4}\right]}=\dfrac{1}{2}\left[\dfrac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{13}}{2}\right)^2}-\dfrac{\left(\frac{3}{2}\right)\left(\frac{\sqrt{13}}{2}\right)}{\left(\frac{\sqrt{13}}{2}\right)\left[\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{13}}{2}\right)^2\right]}\right]$

$$\mathcal{L}^{-1}\{F(s)\}(t)=\frac{1}{2}\left[e^{-\frac{t}{2}}\cdot\cos\left(\frac{\sqrt{13}}{2}t\right)-\frac{3}{\sqrt{13}}\cdot e^{-\frac{t}{2}}\cdot\sin\left(\frac{\sqrt{13}}{2}t\right)\right]$$

$$=\boxed{\frac{1}{2}e^{-\frac{t}{2}}\cdot\cos\left(\frac{\sqrt{13}}{2}t\right)-\frac{3}{2\sqrt{13}}e^{-\frac{t}{2}}\cdot\sin\left(\frac{\sqrt{13}}{2}t\right)}$$

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(8pts.) b.) $G(s)=\dfrac{3}{(2s+5)^3}=\dfrac{3}{2^3\left(s+\frac{5}{2}\right)^3}=\dfrac{3}{8}\cdot\dfrac{2!}{2\left(s+\frac{5}{2}\right)^3}=\dfrac{3}{16}\cdot\dfrac{2!}{\left(s+\frac{5}{2}\right)^3}$

$$\Rightarrow \mathcal{L}^{-1}\{G(s)\}(t)=\boxed{\frac{3}{16}e^{-\frac{5}{2}t}\cdot t^2}$$

(8pts.) c-) $H(s)=\dfrac{1}{(s^2+1)^2}=\dfrac{1}{s^2+1}\cdot\dfrac{1}{s^2+1}$

$$\mathcal{L}^{-1}\{H(s)\}(t)=\left(\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t)\right)*\left(\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t)\right)=\sin t * \sin t$$

$$=\int_0^t \sin v\cdot\sin(t-v)\,dv=\int_0^t\frac{1}{2}\left[\cos(2v-t)-\cos t\right]dv$$

$$=\frac{1}{2}\left[\frac{\sin(2v-t)}{2}-(\cos t)(v)\Big|_0^t\right]=\frac{1}{2}\Bigg[\underbrace{\frac{\sin t}{2}-\frac{\sin(-t)}{2}}_{\sin t}-t(\cos t)\Bigg]$$

$$=\boxed{\frac{1}{2}\left[\sin t-t\cos t\right]}$$