SOLUTIONS

5-Differential Equations — Fall 2018 — Midterm II

or: B. Kaymakçalan

Please show as much work as possible to get full or partial credit. Please circle your answers. Total: 100+20 points. Time allowed 90 minutes

****

1) Consider the equation $xy'' + ay' + \frac{b}{x}y = 0, \quad x > 0.$

(5 pts.) **(a)** Find a relation between $\boldsymbol{a}$ and $\boldsymbol{b}$ for which $y_1 = \frac{1}{x}$ is a solution of the given differential equation.

$y_1 = \frac{1}{x}, \quad y_1' = -\frac{1}{x^2}, \quad y_1'' = \frac{2}{x^3} \Rightarrow$

$x\left[\frac{2}{x^3}\right] + a\left[-\frac{1}{x^2}\right] + \left(\frac{b}{x}\right)\left[\frac{1}{x}\right] = \frac{2}{x^2} - \frac{a}{x^2} + \frac{b}{x^2} = \frac{2-(a-b)}{x^2} = 0$

$\Rightarrow \boxed{a - b = 2}$

(6 pts.) **(b)** Find a second solution y_2 when $\boldsymbol{a} = \mathbf{3}$ and $\boldsymbol{b} = \mathbf{1}$ such that y_1 and y_2 form two linearly independent solutions of the equation. (Hint: Let $y(x) = v(x)\frac{1}{x}$)

$\left.\begin{matrix} a=3 \\ b=1 \end{matrix}\right\} \Rightarrow \boxed{xy'' + 3y' + \frac{1}{x}y = 0} \Rightarrow y(x) = v\cdot\frac{1}{x} \Rightarrow y' = v'\cdot\frac{1}{x} - v\cdot\frac{1}{x^2}$

$\Rightarrow y'' = v''\cdot\frac{1}{x} - v'\cdot\frac{2}{x^2} + v\cdot\frac{2}{x^3}$

into the D.E. $\Rightarrow x\left[v''\frac{1}{x} - \frac{2v'}{x^2} + \frac{2v}{x^3}\right] + 3\left[\frac{v'}{x} - \frac{v}{x^2}\right] + \frac{1}{x}\left[v\cdot\frac{1}{x}\right] = 0$

$\Rightarrow v''[1] + v'\left[\frac{3}{x} - \frac{2}{x}\right] + v\underbrace{\left[\frac{2}{x^2} - \frac{3}{x^2} + \frac{1}{x^2}\right]}_{=0} = 0 \Rightarrow \boxed{v'' + \frac{1}{x}v' = 0} \Rightarrow$

Letting $\left.\begin{matrix} w=v' \\ w'=v'' \end{matrix}\right\}$ (by reduction of order)

$w' + \frac{1}{x}w = 0 \Rightarrow \frac{dw}{w} = -\frac{1}{x}dx \Rightarrow \ln w = -\ln x + \ln c_1$

$\Rightarrow \ln w = \ln\left(\frac{c_1}{x}\right) \Rightarrow w = v' = \frac{c_1}{x} \Rightarrow \boxed{v = c_1 \ln x + c_2}$

$y(x) = (c_1 \ln x + c_2)\left(\frac{1}{x}\right) = c_2\underbrace{\left(\frac{1}{x}\right)}_{y_1(x)} + c_1\underbrace{\left(\frac{1}{x}\cdot\ln x\right)}_{y_2(x)}$

$\Rightarrow \boxed{y_2(x) = \frac{1}{x}\cdot\ln x}$

3) **(21 pts.)** Find a <u>general solution</u> to the equation $y''-6y'+9y=x^{-3}e^{3x}$, $x>0$.
(use VP method)

y_c: $r^2-6r+9=0 \Rightarrow (r-3)^2=0 \Rightarrow r_1=r_2=3 \Rightarrow y_c(x)=c_1\underbrace{e^{3x}}_{y_1(x)}+c_2\underbrace{xe^{3x}}_{y_2(x)}$

$y_p(x)=v_1(x)\cdot\underbrace{(e^{3x})}_{y_1(x)}+v_2(x)\cdot\underbrace{(xe^{3x})}_{y_2(x)}$

$W(y_1,y_2)(x)=\begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x}+3xe^{3x}\end{vmatrix}$

$= e^{6x}+\cancel{3xe^{6x}}-\cancel{3xe^{6x}}=e^{6x}\neq 0$

$\Rightarrow \boxed{W(y_1,y_2)(x)=e^{6x}}$

$v_1'(x)=\dfrac{\begin{vmatrix} 0 & xe^{3x} \\ x^{-3}e^{3x} & e^{3x}+3xe^{3x}\end{vmatrix}}{e^{6x}}$

$=\dfrac{-x^{-2}\cdot\cancel{e^{6x}}}{\cancel{e^{6x}}}=-\dfrac{1}{x^2}$

$\Rightarrow \boxed{v_1(x)}=\int -\dfrac{1}{x^2}\,dx=\boxed{\dfrac{1}{x}}$

$v_2'(x)=\dfrac{\begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & x^{-3}\cdot e^{3x}\end{vmatrix}}{e^{6x}}=\dfrac{x^{-3}\cdot\cancel{e^{6x}}}{\cancel{e^{6x}}}=\dfrac{1}{x^3}\Rightarrow\boxed{v_2(x)}=\int\dfrac{1}{x^3}\,dx=\boxed{\dfrac{-1}{2x^2}}$

$\Rightarrow y_p(x)=\left(\dfrac{1}{x}\right)\left(e^{3x}\right)+\left(-\dfrac{1}{2x^2}\right)\left(xe^{3x}\right)=\dfrac{e^{3x}}{x}-\dfrac{e^{3x}}{2x}=\boxed{\dfrac{e^{3x}}{2x}}$

$\boxed{y_g(x)}=y_c(x)+y_p(x)=\boxed{c_1e^{3x}+c_2xe^{3x}+\dfrac{e^{3x}}{2x}}$

(8 pts.) 4) Use the definition of the Laplace transform to determine $\mathcal{L}\{f(t)\}(s)$ for

$$f(t) = \begin{cases} e^{-2t}, & 0 < t < 3 \\ 3, & t > 3 \end{cases}$$

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^3 e^{-st} \cdot e^{-2t}\,dt + \int_3^\infty e^{-st} \cdot 3\,dt$$

$$= \frac{e^{-(s+2)t}}{-(s+2)}\Bigg|_0^3 + \lim_{N\to\infty} \left(\frac{3 \cdot e^{-st}}{-s}\right)\Bigg|_3^N = -\frac{1}{(s+2)}\left[e^{-(s+2)3} - \underbrace{e^{0}}_{=1}\right] - \frac{3}{s}\left[\cancel{e^{-s\infty}}^{\,0\ (s>0)} - e^{-3s}\right]$$

$$= \boxed{\frac{1}{s+2} - \frac{e^{-3(s+2)}}{s+2} + \frac{3}{s}e^{-3s}} \quad (s>0)$$

(12 pts.) 5) Evaluate the Laplace transform of the following:

$$\mathcal{L}\{e^{3t}\sin(6t) - t^3e^{5t} + e^{-2t}\cos(\sqrt{5}\,t)\}(s) =$$

$$= \mathcal{L}\{e^{3t} \cdot \sin 6t\}(s) - \mathcal{L}\{t^3e^{5t}\}(s) + \mathcal{L}\{e^{-2t} \cdot \cos(\sqrt{5}t)\}(s)$$

$$= \frac{6}{(s-3)^2+6^2} - \frac{3!}{(s-5)^4} + \frac{(s+2)}{(s+2)^2+(\sqrt{5})^2}$$

$$= \boxed{\frac{6}{(s-3)^2+36} - \frac{6}{(s-5)^4} + \frac{(s+2)}{(s+2)^2+5}}$$

(12 pts.) 6) Determine the inverse Laplace transform $f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$ for

$$s^2F(s) + sF(s) - 6F(s) = \frac{s^2+4}{s^2+s}$$

$$F(s)\underbrace{\left[s^2+s-6\right]}_{(s+3)(s-2)} = \frac{s^2+4}{s^2+s} \Rightarrow F(s) = \frac{1}{(s+3)(s-2)} \cdot \frac{s^2+4}{s(s+1)}$$

$$= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3} + \frac{D}{s-2}$$

$$\Rightarrow F(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3} + \frac{D}{s-2} = \frac{s^2+4}{s(s+1)(s+3)(s-2)}$$

$$= \frac{A(s+1)(s+3)(s-2) + Bs(s+3)(s-2) + Cs(s+1)(s-2) + Ds(s+1)(s+3)}{s(s+1)(s+3)(s-2)}$$

$s=0 \Rightarrow 4 = -6A \Rightarrow \boxed{A = -\frac{2}{3}}$

$s=-1 \Rightarrow 5 = 6B \Rightarrow \boxed{B = \frac{5}{6}}$

$s=2 \Rightarrow 8 = 30D \Rightarrow \boxed{D = \frac{4}{15}}$

$s=-3 \Rightarrow 13 = -30C \Rightarrow \boxed{C = -\frac{13}{30}}$

$$\Rightarrow F(s) = \left(-\frac{2}{3}\right)\frac{1}{s} + \left(\frac{5}{6}\right)\frac{1}{s+1} + \left(\frac{-13}{30}\right)\frac{1}{s+3} + \left(\frac{4}{15}\right)\cdot\frac{1}{s-2}$$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\{F(s)\}(t) = \left(-\frac{2}{3}\right)\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) + \left(\frac{5}{6}\right)\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) - \frac{13}{30}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}(t) + \left(\frac{4}{15}\right)\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}(t)$$

$$\Rightarrow \boxed{f(t) = -\frac{2}{3} + \frac{5}{6}e^{-t} - \frac{13}{30}e^{-3t} + \frac{4}{15}e^{2t}}$$

(17 pts.) 7) Solve the below integro-differential equation using Laplace transform:

$$y'(t) + y(t) - \underbrace{\int_0^t y(v)\sin(t-v)\,dv}_{y(t)*\sin t} = -\sin t, \qquad y(0) = 1$$

$\mathcal{L}\{y(t)\}(s) = Y(s)$

$\mathcal{L}\{\sin t\}(s) = \frac{1}{s^2+1}$

$$\mathcal{L}\{y'(t)\}(s) + \mathcal{L}\{y(t)\}(s) - \mathcal{L}\{y(t)*\sin t\}(s) = -\mathcal{L}\{\sin t\}(s)$$

$$[sY(s) - \underbrace{y(0)}_{1}] + Y(s) - Y(s)\cdot\frac{1}{s^2+1} = -\frac{1}{s^2+1}$$

$$Y(s)\underbrace{\left[s+1-\frac{1}{s^2+1}\right]}_{\frac{s^3+s^2+s+\cancel{1}-\cancel{1}}{s^2+1}} = -\frac{1}{s^2+1}+1 = \frac{s^2+\cancel{1}-\cancel{1}}{s^2+1} = \frac{s^2}{s^2+1}$$

$$Y(s) = \frac{\cancel{s^2+1}}{\cancel{s}(s^2+s+1)}\cdot\frac{s^{\cancel{2}}}{\cancel{s^2+1}} = \frac{s}{s^2+s+1} = \frac{\left(s+\frac{1}{2}\right)-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$Y(s) = \frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)} - \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\sqrt{3}}\cdot\frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\boxed{y(t) = \mathcal{L}^{-1}\{Y(s)\}(t) = e^{-\frac{t}{2}}\cdot\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}e^{-\frac{t}{2}}\cdot\sin\left(\frac{\sqrt{3}}{2}t\right)}$$

Extra Credit Problems: (10 points each)

1) Find the general solution of

$$\mathbf{(x^2-2x)y''(x)-(x^2-2)y'(x)+2(x-1)y(x)=3(x^2-2x)^2e^x}$$

given that $\mathbf{e^x}$ is a solution of the corresponding homogeneous equation.

(Hint: Use Reduction of Order method.)

a) $y=v.e^x \Rightarrow y'=v'e^x+v.e^x \Rightarrow y''=v''e^x+2v'e^x+ve^x$

$(x^2-2x)(v''+2v'+v)e^x-(x^2-2)(v'+v)e^x+2(x-1)ve^x=0$

$\underset{\neq 0}{e^x}\left[v''(x^2-2x)+v'(2x^2-4x-x^2+2)+v(\cancel{x^2}-\cancel{2x}-\cancel{x^2}+\cancel{2}+\cancel{2x}-\cancel{2})\right]=0$

$\Rightarrow v''(x^2-2x)+v'(x^2-4x+2)=0$

$v''+\dfrac{(x^2-4x+2)}{(x^2-2x)}v'=0 \Rightarrow \left.\begin{array}{l} w=v' \\ w'=v'' \end{array}\right\} \Rightarrow \omega'+\dfrac{(x^2-2x)+(2-2x)}{(x^2-2x)}\omega=0$

$\Rightarrow -\displaystyle\int\frac{dw}{\omega}=\int\left[\frac{(x^2-2x)}{(x^2-2x)}-\frac{(2x-2)}{(x^2-2x)}\right]dx \Rightarrow -\ln|\omega|=x-\ln(x^2-2x)$

$\Rightarrow \ln\left|\dfrac{x^2-2x}{\omega}\right|=x \Rightarrow \dfrac{x^2-2x}{\omega}=e^x \Rightarrow \underset{v'}{\omega}=e^{-x}(x^2-2x)$

$\Rightarrow v(x)=\displaystyle\int e^{-x}(x^2-2x)\,dx = -x^2.e^{-x}$

integration by parts (twice)

$y_2(x)=v.e^x=(-x^2e^{-x})(e^x)=\boxed{-x^2}$

$y_c=c_1y_1(x)+c_2y_2(x)=c_1e^x+c_2(-x^2)=\boxed{Ae^x+Bx^2}$

$\Rightarrow y_p(x)=v_1(x)\underbrace{(e^x)}_{y_1}+v_2(x)\underbrace{(x^2)}_{y_2}$; $\boxed{W(e^x,x^2)=\begin{vmatrix} e^x & x^2 \\ e^x & 2x \end{vmatrix}=\boxed{e^x(2x-x^2)}}$

Using VP method we find $y_p(x)=(-x^3)e^x+(-3e^x).x^2=\boxed{-x^2e^x(3+x)}$

$\Rightarrow \boxed{y_g=Ae^x+Bx^2+\boxed{-x^2e^x(3+x)}}$

2) Find the general solution to the equation $\mathbf{y''} = \mathbf{5x^{-1}y' - 13x^{-2}y},\ \mathbf{x > 0}$.
(Hint: Can you recognize the type of this equation?)

$y'' = 5x^{-1}y' - 13x^{-2}y \Rightarrow x^2y'' - 5xy' + 13y = 0 \rightarrow$ Cauchy-Euler eqn.

$a=1 \quad b=-5 \quad c=13$

Let $x = e^t \Rightarrow t = \ln x$

$\Rightarrow a\,y''(t) + (b-a)\,y'(t) + c\,y(t) = 0 \Rightarrow y''(t) + (-5-1)y'(t) + 13y(t) = 0$

$\Rightarrow y''(t) - 6y'(t) + 13y(t) = 0 \Rightarrow r^2 - 6r + 13 = 0$

$(r-3)^2 + 4 = 0$

$r - 3 = \mp 2i \Rightarrow \boxed{r = 3 \mp 2i}$

$\boxed{y(t) = c_1 e^{3t} \cdot \cos(2t) + c_2 e^{3t} \cdot \sin(2t)}$, $t = \ln x \Rightarrow$

$\Rightarrow y(x) = c_1 x^3 \cdot \cos(2\ln x) + c_2 x^3 \cdot \sin(2\ln x)$

$\Rightarrow \boxed{y(x) = c_1 x^3 \cdot \cos(\ln x^2) + c_2 x^3 \cdot \sin(\ln x^2)}$