ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 245 - Differential Equations

2019-2020 Fall

SECOND MIDTERM EXAMINATION

23.12.2019, 17:30

SOLUTIONS

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR: B.K.

DURATION: 90 minutes

Question	Grade	Out of
1		24
2		12
3		16
4		16
5		16
6		16
Total		100

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 6 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

1. Find the form of the particular solution for the method of undetermined coefficients. DO NOT EVALUATE THE COEFFICIENTS.

(8 pts.) **a)** $y'' + 2y' + 2y = 8\,t^3 e^{-t} \sin t$

$y_h(t)$: $r^2 + 2r + 2 = 0 \Rightarrow (r+1)^2 + 1 = 0 \Rightarrow r + 1 = \mp i$

$r = -1 \mp i$

$y_h(t) = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t$

$$\boxed{\begin{aligned} y_p(t) &= t(At^3 + Bt^2 + Ct + D)e^{-t} \cos t \\ &+ t(Et^3 + Ft^2 + Gt + H)e^{-t} \sin t \end{aligned}}$$

(8 pts.) **b)** $y'' - y' - 12y = 2\,t^6 e^{-3t}$

$y_h(t)$: $r^2 - r - 12 = 0 \Rightarrow (r-4)(r+3) = 0 \Rightarrow r_1 = 4,\ r_2 = -3$

$y_h(t) = C_1 e^{4t} + C_2 e^{-3t}$

$$\boxed{y_p(t) = t(At^6 + Bt^5 + Ct^4 + Dt^3 + Et^2 + Ft + G)e^{-3t}}$$

(8 pts.) **c)** $y'' + 6y' + 9y = \overbrace{5e^{-3t} + (t^2 - 3)e^{-3t}}^{t^2 e^{-3t} + 2e^{-3t} = (t^2+2)e^{-3t}}$

$y_h(t)$: $r^2 + 6r + 9 = 0 \Rightarrow (r+3)^2 = 0 \Rightarrow r_1 = r_2 = -3$

$y_h(t) = c_1 e^{-3t} + c_2 t e^{-3t}$

$$\boxed{y_p(t) = t^2(At^2 + Bt + C)e^{-3t}}$$

(12 pts.) **2.** Find a <u>general solution</u> to the equation $y'' - 6y' + 9y = x^{-3}e^{3x}$, $x > 0$.

$$y_h(x):\ r^2 - 6r + 9 = 0 \Rightarrow (r-3)^2 = 0 \Rightarrow y_h(x) = C_1 \underbrace{e^{3x}}_{y_1} + C_2 \underbrace{x e^{3x}}_{y_2}$$
$$(r_1 = r_2 = 3)$$

By V.P. method: $y_p(x) = v_1(x) \cdot y_1(x) + v_2(x) \cdot y_2(x)$

where: $W(y_1, y_2)(x) = \begin{vmatrix} e^{3x} & x e^{3x} \\ 3e^{3x} & e^{3x} + 3x e^{3x} \end{vmatrix} = e^{6x} \neq 0$

$$v_1(x) = -\int \frac{x^{-3} \cdot e^{3x} \cdot x e^{3x}}{W(y_1, y_2)(x)}\,dx = -\int \frac{x^{-2} \cdot \cancel{e^{6x}}}{\cancel{e^{6x}}}\,dx = \boxed{\frac{1}{x}}$$

$$v_2(x) = \int \frac{x^{-3} \cdot e^{3x} \cdot e^{3x}}{e^{6x}}\,dx = \int x^{-3}\,dx = \boxed{\frac{-1}{2x^2}}$$

$$\Rightarrow y_p(x) = \left(\frac{1}{x}\right)\left(e^{3x}\right) + \left(\frac{-1}{2x^2}\right)\left(x e^{3x}\right) = \boxed{\frac{e^{3x}}{2x}}$$

$$\Rightarrow y_g = y_h + y_p = \boxed{C_1 e^{3x} + C_2 x e^{3x} + \frac{e^{3x}}{2x}}$$

(16 pts.) **3. a)** If $y_1(x) = xe^x$ is a solution of $y''-2y'+y=0$, find a second linearly independent solution $y_2(x)$ using the method of reduction of order (use only this method!).

$y = v(x)y_1(x) = v \cdot xe^x \Rightarrow y' = v'xe^x + ve^x + vxe^x$

$y'' = v''xe^x + v'e^x + v'xe^x + v'e^x + ve^x + v'xe^x + ve^x + vxe^x$

$y'' = e^x[v''x + 2v' + 2v'x + 2v + vx]$

d.e.: $y''-2y'+y = \cancel{e^x}[v''x + 2v' + \cancel{2v'x} + \cancel{2v} + \cancel{vx} - \cancel{2v'x} - \cancel{2v} - \cancel{2vx} + \cancel{vx}] = 0$

$\Rightarrow v''x + 2v' = 0 \Rightarrow v'' + \frac{2}{x}v' = 0 \Rightarrow \omega' + \frac{2}{x}\omega = 0 \Rightarrow \frac{d\omega}{dx} = -\frac{2}{x}\omega$

Let $v' = \omega$

$\Rightarrow \frac{d\omega}{\omega} = -\frac{2}{x}dx$

$\ln\omega = -2\ln x + \ln c_1$

$v' = \omega = \frac{c_1}{x^2}$

$\Rightarrow v(x) = -\frac{c_1}{x} + c_2$

$\Rightarrow y = \left(-\frac{c_1}{x} + c_2\right)(xe^x) = -c_1\underbrace{e^x}_{y_2} + c_2\underbrace{xe^x}_{y_1}$

$\Rightarrow \boxed{y_2(x) = e^x}$

$\boxed{y_h(x) = Ae^x + Bxe^x}$

$W(y_1, y_2) = \begin{vmatrix} xe^x & e^x \\ e^x + xe^x & e^x \end{vmatrix} = -e^{2x} \neq 0$

b) Using the result of part a), find the general solution of

$$y'' - 2y' + y = \frac{e^x}{x} \quad (x > 0)$$

$y_p = v_1(x)\underbrace{xe^x}_{y_1} + v_2(x)\underbrace{e^x}_{y_2}$

$v_1(x) = \int \frac{-\frac{e^x}{x}\cancel{e^x}}{-\cancel{e^{2x}}}dx = \int \frac{1}{x}dx = \ln|x| = \boxed{\ln x}$

$v_2(x) = \int \frac{\frac{e^x}{x}xe^x}{-e^{2x}}dx = \int -dx = \boxed{-x}$

$y_p = (\ln x)(xe^x) + (-x)(e^x)$

$\Rightarrow y_g = Ae^x + Bxe^x + xe^x\ln x - xe^x = \boxed{\underbrace{[Ae^x + Cxe^x]}_{y_h} + \underbrace{xe^x\ln x}_{y_p}}$

$(C = B - 1)$

(16 pts.) **4.** Find the general solution to the equation

$$y'' = 5x^{-1}y' - 13x^{-2}y, \quad x > 0$$

(Hint: Can you recognize the type of this equation?)

$x^2y'' - 5xy' + 13y = 0$ (Cauchy-Euler eqn.)

$x = e^t \Rightarrow y'' + (-5-1)y' + 13y = 0$

$r^2 - 6r + 13 = 0 \Rightarrow (r-3)^2 + 4 = 0 \Rightarrow r = 3 \mp 2i$

$\Rightarrow y(t) = C_1 e^{3t}\cos(2t) + C_2 e^{3t}\sin(2t)$

$\left.\begin{array}{l} x = e^t \\ t = \ln x \end{array}\right\}$ $\boxed{y(x) = C_1 x^3 \cdot \cos(2\ln x) + C_2 x^3 \cdot \sin(2\ln x)}$

(16 pts.) **5.** Using Laplace transform, solve the I.V.P. ;

$$y'' - 2y' + y = 4\sin x \cos x\, e^x \; ; \quad y(0) = 0, \quad y'(0) = 1$$

($\underline{\text{Hint}}$: $2\sin x \cos x = \sin(2x)$)

$$y'' - 2y' + y = 2(\sin 2x)(e^x)$$

$$\mathcal{L}\{y''\}(s) - 2\mathcal{L}\{y'\}(s) + \mathcal{L}\{y\}(s) = 2\mathcal{L}\{e^x . \sin 2x\}(s)$$

$$\left[s^2 Y(s) - s\underbrace{\cancel{y(0)}}_{0} - \underbrace{y'(0)}_{1}\right] - 2\left[sY(s) - \underbrace{\cancel{y(0)}}_{0}\right] + Y(s) = 2\,\frac{2}{(s-1)^2+4}$$

$$Y(s)\underbrace{\left[s^2 - 2s + 1\right]}_{(s-1)^2} = 1 + \frac{4}{(s-1)^2+4} \Rightarrow Y(s) = \frac{1}{(s-1)^2} + 2 \cdot \frac{1}{(s-1)^2} \cdot \frac{2}{(s-1)^2+4}$$

$$\Rightarrow \boxed{y(x) = \mathcal{L}^{-1}\{Y(s)\}(x) = xe^x + 2\left(xe^x * e^x \sin(2x)\right)}$$

$$xe^x * e^x \sin(2x) = \int_0^x (x-v)\underbrace{e^{(x-v)} \cdot e^{v}}_{e^x} . \sin(2v)\, dv$$

$$= xe^x \int_0^x \sin(2v)\, dv - e^x \int_0^x v . \sin(2v)\, dv$$

$$= \frac{xe^x}{2} - \frac{e^x}{4}\sin(2x)$$

$$\Rightarrow \quad y(x) = xe^x + xe^x - \frac{e^x . \sin(2x)}{2}$$

$$= \boxed{2xe^x - \frac{e^x . \sin(2x)}{2}}$$

7.

$\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$

(16 pts.) **6.** Solve the following integro-differential equation;

$$y' = \sin t + \int_0^t y(t-v)\cos v \, dv \ , \quad y(0) = 0$$

$$\mathcal{L}\{y'(t)\}(s) = \mathcal{L}\{\sin t\}(s) + \underbrace{\mathcal{L}\{y(t) * \cos t\}(s)}_{Y(s)\cdot \frac{s}{s^2+1}}$$

$$[sY(s) - \cancelto{0}{y(0)}] = \frac{1}{s^2+1} + Y(s)\cdot\frac{s}{s^2+1}$$

$$Y(s)\underbrace{\left[s - \frac{s}{s^2+1}\right]}_{\frac{s^3+\cancel{s}-\cancel{s}}{s^2+1}} = \frac{1}{s^2+1} \Rightarrow Y(s) = \frac{\cancel{s^2+1}}{s^3}\cdot\frac{1}{\cancel{s^2+1}}$$

$$\Rightarrow Y(s) = \frac{1}{s^3} \Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\}(t) = \boxed{\frac{t^2}{2}}$$