ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 258 - Introduction to Differential Equations

FIRST MIDTERM EXAMINATION

20.03.2017

STUDENT NUMBER: ANSWER KEY

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR:

DURATION: 100 minutes

Question	Grade	Out of
1		17
2		17
3		17
4		17
5		17
6		17
Total		102

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 6 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

Question 1. Solve the equation $2 + \frac{dx}{dy} = y - x$.

I. $\frac{dx}{dy} = \underbrace{y-x-2}_{G(y-x)}$ $\quad y-x=u \quad 1-\frac{dx}{dy} = \frac{du}{dy} \quad \frac{dx}{dy} = 1-\frac{du}{dy} \quad 1-\frac{du}{dy} = u-2$

$3-u = \frac{du}{dy} \quad \frac{du}{3-u} = dy \quad -\ln|3-u| = y + \ln c_1 \quad \frac{1}{3-u} = \tilde{c}_1 e^y \quad c = (3-u)e^y$

$\boxed{(3-y+x)e^y = c}$

II. $\underbrace{1}_{M(x,y)}.dx - \underbrace{(y-x-2)}_{N(x,y)}dy = 0 \quad M_y = 0 \neq N_x = 1$ (not exact)

$\frac{N_x - M_y}{M} = 1 = f(y) \quad \frac{d\mu}{\mu} = dy \quad \ln|\mu| = y + \ln c \quad \mu(y) = e^y$

$e^y dy - e^y(y-x-2)dy = 0$ (exact)

$F_x = e^y$

$F_y = -ye^y + xe^y + 2e^y$

$\begin{cases} F(x,y) = xe^y + \varphi(y) \\ xe^y + \varphi'(y) = -ye^y + xe^y + 2e^y \end{cases} \quad \varphi'(y) = -ye^y + 2e^y$

$\varphi(y) = -\int y\,d(e^y) + 2e^y + c_1 \quad \varphi(y) = -\left[ye^y - \int e^y dy\right] + 2e^y + c_1$

$\varphi(y) = -ye^y + \underbrace{e^y + 2e^y}_{} + c_1 \quad F(x,y) = xe^y - ye^y + 3e^y + c_1$

$xe^y - ye^y + 3e^y = c \quad \boxed{(3-y+x)e^y = c}$

III. $\frac{dy}{dx} + \underbrace{1}_{P(y)}.x = \underbrace{y-2}_{Q(y)}$ (linear) $\quad \mu(y) = e^{\int dy} = e^y$

$e^y.x = \int e^y(y-2)dy = \int (y-2)d(e^y) = (y-2)e^y - \int e^y dy = (y-2)e^y - e^y + c$

$x = y-2-1+ce^{-y} \quad (x-y+3) = ce^{-y} \quad \boxed{(x-y+3)e^y = c}$

Question 2. It is known that the equation

$$(asiny + y^2e^x)dx + (xcosy + bye^x)dy = 0$$

is exact. Find a and b and then solve the equation.

$$\underbrace{(a\sin y + y^2e^x)}_{M(x,y)}dx + \underbrace{(x\cos y + bye^x)}_{N(x,y)}dy = 0 \quad (\text{exact})$$

$$M_y = a\cos y + 2ye^x = N_x = \cos y + bye^x \Rightarrow a = 1,\ b = 2$$

$$(\sin y + y^2e^x)dx + (x\cos y + 2ye^x)dy = 0 \quad (\text{exact})$$

$$F_x = \sin y + y^2e^x \qquad F(x,y) = x\sin y + y^2e^x + \varphi(y)$$

$$F_y = x\cos y + 2ye^x \qquad x\cos y + 2ye^x + \varphi'(y) = x\cos y + 2ye^x \Rightarrow \varphi(y) = c_1$$

$$\boxed{x\sin y + y^2e^x = c}$$

Question 3. Solve the equation $x^{-2}y' = y^2 - 2x^{-2} - 2x^{-2}y^2 + 1$.

I. $x^{-2}\dfrac{dy}{dx} = y^2 - 2x^{-2} - 2x^{-2}y^2 + 1 \qquad \dfrac{dy}{dx} = x^2y^2 - 2 - 2y^2 + x^2$

$$\frac{dy}{dx} = x^2(y^2+1) - 2(y^2+1) \qquad \frac{dy}{dx} = (x^2-2)(y^2+1)$$

$$\frac{dy}{1+y^2} = (x^2-2)\,dx \qquad \arctan y = \frac{x^3}{3} - 2x + c \qquad \boxed{y = \tan\left(\frac{x^3}{3} - 2x + c\right)}$$

II. $\underbrace{(x^2-2)}_{M(x,y)}dx - \underbrace{\frac{1}{1+y^2}}_{N(x,y)}dy = 0$

$M_y = 0 = N_x$ (exact)

$$\left.\begin{aligned} F_x &= x^2 - 2 \\ F_y &= -\frac{1}{1+y^2} \end{aligned}\right\} \qquad \begin{aligned} &F(x,y) = \frac{x^3}{3} - 2x + \varphi(y) \\ &\varphi'(y) = -\frac{1}{1+y^2} \qquad \varphi(y) = -\arctan y + c_1 \end{aligned}$$

$$F(x,y) = \frac{x^3}{3} - 2x - \arctan y + c_1$$

$$\frac{x^3}{3} - 2x - \arctan y = \tilde{c} \qquad \arctan y = \frac{x^3}{3} - 2x + c \qquad \boxed{y = \tan\left(\frac{x^3}{3} - 2x + c\right)}$$

Question 4. Solve the IVP

$$x^2y' + xy = xy^2, y(1) = \frac{1}{2}.$$

I. $\frac{dy}{dx} + \underbrace{\frac{1}{x}}_{P(x)} y = \underbrace{\frac{1}{x}}_{Q(x)} y^{2}$ $2 = n$ (Bernoulli)

$v = y^{1-2} = y^{-1}$ $\quad \frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$ $\quad \frac{dy}{dx} = -y^2\frac{dv}{dx}$ $\quad -y^2\frac{dv}{dx} + \frac{1}{x}y = \frac{1}{x}y^2$

$\frac{dv}{dx} - \underbrace{\frac{1}{x}}_{\tilde{P}(x)} v = \underbrace{-\frac{1}{x}}_{\tilde{Q}(x)}$ (linear) $\quad \mu(x) = e^{-\int \frac{1}{x}dx} = e^{-\ln|x|} = x^{-1}$

$\frac{1}{x}v = \int \frac{1}{x}\left(-\frac{1}{x}\right)dx$ $\quad \frac{1}{x}v = \frac{1}{x} + c$ $\quad v = 1 + cx$ $\quad y^{-1} = 1 + cx$

$y(1) = \frac{1}{2} \Rightarrow 2 = 1 + c \Rightarrow c = 1$ $\quad \boxed{y^{-1} = 1 + x}$

II. $x^2\frac{dy}{dx} = xy^2 - xy$ $\quad x\,dy = (y^2 - y)dx$

$\underbrace{(y^2-y)}_{M(x,y)}dx \underbrace{-x}_{N(x,y)}dy = 0$ $\quad M_y = 2y - 1 \neq N_x = -1$ (not exact)

$\frac{N_x - M_y}{M} = \frac{-1-2y+1}{y(y-1)} = \frac{-2}{y-1} = f(y)$ $\quad \frac{d\mu}{\mu} = \frac{-2}{y-1}dy$ $\quad \ln|\mu| = -2\ln|y-1| + \ln c_1$

$\mu(y) = \frac{1}{(y-1)^2}$

$\frac{y}{y-1}dx - \frac{x}{(y-1)^2}dy = 0$ (exact)

$F_x = 1 + \frac{1}{y-1}$, $F_y = -\frac{x}{(y-1)^2}$ $\quad F(x,y) = x + \frac{x}{y-1} + \varphi(y)$ $\quad -\frac{x}{(y-1)^2} + \varphi'(y) = -\frac{x}{(y-1)^2}$

$\Rightarrow \varphi(y) = c_1$ $\quad x + \frac{x}{y-1} = c$ $\quad y(1) = \frac{1}{2} \Rightarrow 1 + (-2) = c$

$c = -1$ $\quad x + \frac{x}{y-1} = -1$ $\quad \frac{xy - x + x}{y-1} = -1$ $\quad xy = -y + 1$ $\quad x = -1 + y^{-1}$

$\boxed{y^{-1} = 1 + x}$

Question 5. Find an integrating factor of the form $x^m y^n$ for the equation

$$(4xy^2 + 6y)dx + (5x^2y + 8x)dy = 0$$

and then solve the equation.

$$\underbrace{\left(4x^{m+1}y^{n+2} + 6x^{m}y^{n+1}\right)}_{M(x,y)}dx + \underbrace{\left(5x^{m+2}y^{n+1} + 8x^{m+1}y^{n}\right)}_{N(x,y)}dy = 0$$

$$\left.\begin{aligned} M_y &= 4(n+2)x^{m+1}y^{n+1} + 6(n+1)x^m y^n \\ N_x &= 5(m+2)x^{m+1}y^{n+1} + 8(m+1)x^m y^n \end{aligned}\right\} \quad \left.\begin{aligned} 4(n+2) &= 5(m+2) \\ 6(n+1) &= 8(m+1) \end{aligned}\right\} \quad \begin{aligned} 4n+8 &= 5m+10 \\ 3n+3 &= 4m+4 \end{aligned}$$

$$m = 2, \quad n = 3$$

$$\mu(x,y) = x^2 y^3$$

$$(4x^3y^5 + 6x^2y^4)dx + (5x^4y^4 + 8x^3y^3)dy = 0 \quad \text{(exact)}$$

$$\left.\begin{aligned} F_x &= 4x^3y^5 + 6x^2y^4 \\ F_y &= 5x^4y^4 + 8x^3y^3 \end{aligned}\right\} \quad \begin{aligned} &F(x,y) = x^4y^5 + 2x^3y^4 + \varphi(y) \\ &5x^4y^4 + 8x^3y^3 + \varphi'(y) = 5x^4y^4 + 8x^3y^3 \qquad \varphi(y) = c_1 \end{aligned}$$

$$\boxed{x^4y^5 + 2x^3y^4 = C}$$

Question 6. Solve the IVP

$$(4x+3y+1)dx+(x+y+1)dy=0, y(3)=-4.$$

$$\begin{aligned} 4h+3k+1&=0 \\ -3/\quad h+k+1&=0 \end{aligned} \quad \left\{ \begin{aligned} h&=2 \\ k&=-3 \end{aligned} \right\} \quad \begin{aligned} x&=u+2 \Rightarrow dx=du \\ y&=v-3 \Rightarrow dy=dv \end{aligned}$$

$$[4(u+2)+3(v-3)+1]du+[u+2+v-3+1]dv=0$$

$$(4u+3v)du+(u+v)dv=0 \qquad \frac{dv}{du}=-\frac{4u+3v}{u+v} \text{ (homogeneous)}$$

$$\frac{v}{u}=z \Rightarrow v=z.u \qquad \frac{dv}{du}=z+u\frac{dz}{du}$$

$$z+u\frac{dz}{du}=-\frac{4u+3z.u}{u+zu} \qquad z+u\frac{dz}{du}=-\frac{4+3z}{1+z} \qquad u\frac{dz}{du}=\frac{-4-3z-z-z^2}{1+z}$$

$$u\frac{dz}{du}=\frac{-z^2-4z-4}{1+z} \qquad \frac{1+z}{z^2+4z+4}dz=-\frac{du}{u} \qquad \frac{1+z}{(z+2)^2}dz=-\frac{du}{u}$$

$$\frac{z+2-1}{(z+2)^2}dz=\frac{du}{u} \qquad \left[\frac{1}{z+2}-\frac{1}{(z+2)^2}\right]dz=-\frac{du}{u}$$

$$\ln|z+2|+\frac{1}{z+2}=-\ln|u|+\ln c_1 \qquad (z+2)e^{\frac{1}{z+2}}=\frac{c}{u}$$

$$\left(\frac{v}{u}+2\right)e^{\frac{1}{\frac{v}{u}+2}}=\frac{c}{u} \qquad (v+2u)e^{\frac{u}{v+2u}}=c$$

$$[y+3+2(x-2)]e^{\frac{x-2}{y+3+2(x-2)}}=c \qquad (y+2x-1)e^{\frac{x-2}{y+2x-1}}=c$$

$$y(3)=-4 \Rightarrow (-4+6-1)e^{\frac{1}{-4+6-1}}=c \Rightarrow c=e$$

$$\boxed{(y+2x-1)\exp\left(\frac{x-2}{y+2x-1}\right)=e}$$