ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 258 - Introduction to Differential Equations

FIRST MIDTERM EXAMINATION

31.10.2016

Answer KEY

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR:

DURATION: 110 minutes

Question	Grade	Out of
1		20
2		20
3		20
4		20
5		12
6		12
Total		104

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 6 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

Question 1. Solve the equation

$$\frac{dy}{dx} = \frac{x}{x^2+y}.$$

$\frac{dx}{dy} = \frac{x^2+y}{x} = x + \frac{y}{x} \Rightarrow \frac{dx}{dy} - x = y x^{-1}$ Bernoulli in x.

$x\frac{dx}{dy} - x^2 = y$ Let $v = x^2 \Rightarrow \frac{dv}{dy} = 2x\frac{dx}{dy}$

So $\frac{1}{2}\frac{dv}{dy} - v = y$ or $\frac{dv}{dy} - 2v = 2y$

Now $\mu(y) = e^{\int -2\,dy} = e^{-2y}$ is an integrating factor.

$e^{-2y}\frac{dv}{dy} - 2e^{-2y}v = 2ye^{-2y} \Rightarrow \frac{d}{dy}(e^{-2y}v) = 2ye^{-2y}$

so $e^{-2y}v = \int 2ye^{-2y}\,dy = -(y+\frac{1}{2})e^{-2y} + c$

$\therefore v = -(y+\frac{1}{2}) + ce^{2y}$ and hence $\boxed{x^2 = -(y+\frac{1}{2}) + ce^{2y}}$

or

$x\,dx - (x^2+y)\,dy = 0$ clearly the eqn is not exact.

since $M_y = 0$ $N_x = -2x$

$\frac{M_y - N_x}{N} = \frac{+2x}{x^2+y}$ does not depen on x only.

$\frac{N_x - M_y}{M} = \frac{-2x}{x} = -2 \Rightarrow \mu(y) = e^{\int -2\,dy} = e^{-2y}$ is an I.F.

$xe^{-2y}dx - (x^2+y)e^{-2y}dy = 0$ is exact.

$\exists\ F(x,y) = c_1$ such that $\frac{\partial F}{\partial x} = xe^{-2y}$ ---- (1) and $\frac{\partial F}{\partial y} = -x^2e^{-2y} - ye^{-2y}$ ---(2)

(1) $\Rightarrow F(x,y) = \frac{x^2}{2}e^{-2y} + f(y) \Rightarrow \frac{\partial F}{\partial y} = -x^2e^{-2y} + f'(y) \overset{(2)}{=} -x^2e^{-2y} - ye^{-2y}$

so $f'(y) = -ye^{-2y} \Rightarrow f(y) = \frac{1}{2}(y+\frac{1}{2})e^{-2y} + c_2$

So $F(x,y) = \frac{x^2}{2}e^{-2y} + \frac{1}{2}(y+\frac{1}{2})e^{-2y} + c_2 = c_1$

Soln is $\boxed{\frac{x^2}{2}e^{-2y} + \frac{1}{2}(y+\frac{1}{2})e^{-2y} = c}$ where $c = c_1 - c_2$

Question 2. Solve the equation

$$\frac{dy}{dx} = \tan^2(x+y).$$

Let $u = x+y$ $\qquad \dfrac{du}{dx} = 1 + \dfrac{dy}{dx}$

So $\dfrac{du}{dx} - 1 = \tan^2 u$

$$\frac{du}{dx} = \tan^2 u + 1 = \sec^2 u$$

$$\cos^2 u \, du = dx$$

$$\int \frac{1+\cos 2u}{2} \, du = \int dx$$

$$\frac{1}{2} u + \frac{1}{4} \sin 2u = x + c$$

$$\boxed{\frac{1}{2}(x+y) + \frac{1}{4}\sin[2(x+y)] = x + c}$$

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Question 3. Solve the equation $y^3dx = x^3dy - 2x^2ydx$.

$$(y^3+2x^2y)\,dx = x^3\,dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^3+2x^2y}{x^3} = \left(\frac{y}{x}\right)^3 + 2\left(\frac{y}{x}\right)$$ the eqn is hom.

let $v = \frac{y}{x} \Rightarrow y = vx \Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$

$$x\frac{dv}{dx} + v = v^3 + 2v \Rightarrow x\frac{dv}{dx} = v^3 + v$$

$$\frac{dv}{v^3+v} = \frac{dx}{x}$$

$$\frac{1}{v^3+v} = \frac{1}{v(v^2+1)} = \frac{A}{v} + \frac{Bv+C}{v^2+1}$$

$$= \frac{Av^2+A+Bv^2+Cv}{v^2+1}$$

$C=0$ $B=-1$
$A=1$

$$\Rightarrow \int\left(\frac{1}{v} - \frac{v}{v^2+1}\right)dv = \int\frac{dx}{x}$$

$$\ln|v| - \frac{1}{2}\ln(v^2+1) = \ln|x| + \ln c$$

$$\ln\left|\frac{v}{\sqrt{v^2+1}}\right| = \ln c|x|$$

$$\frac{|v|}{\sqrt{v^2+1}} = c|x| \Rightarrow \frac{\frac{y}{|x|}}{\frac{\sqrt{x^2+y^2}}{|x|}} = c|x|$$

$$\boxed{y = c|x|\sqrt{x^2+y^2}} \quad \text{or} \quad \boxed{y = cx\sqrt{x^2+y^2}}$$

Question 4. For the equation

$$y(1+x)dx + x(1-y)dy = 0$$

find an integrating factor of the form $x^m y^n$ and then solve the IVP

$$y(1+x)dx + x(1-y)dy = 0, \ y(1) = 1.$$

$x^m y^{n+1}(1+x)\,dx + x^{m+1} y^n (1-y)\,dy = 0$ is exact.

$$\frac{\partial}{\partial y}\left[x^m y^{n+1}(1+x)\right] = \frac{\partial}{\partial x}\left[x^{m+1} y^n (1-y)\right]$$

$$(n+1)\cancel{x^m y^n}(1+x) = (m+1)\cancel{x^m y^n}(1-y)$$

$$(n+1) + (n+1)x = (m+1) - (m+1)y \quad \text{for all } x, y$$

so $n+1=0$ and $m+1=0$. Therefore $m=n=-1$

Then the eqn becomes:

$$x^{-1}(1+x)dx + y^{-1}(1-y)\,dy = 0 \quad \text{or}$$

$$(x^{-1}+1)\,dx + (y^{-1}-1)\,dy = 0$$

$\exists\ F(x,y) = C_1$ such that $\frac{\partial F}{\partial x} = x^{-1}+1$ ---- (1)

$$\frac{\partial F}{\partial y} = y^{-1}-1 \quad \text{---- (2)}$$

(1) $\Rightarrow$ $F(x,y) = \int x^{-1}+1\,dx = \ln|x| + x + g(y)$

$$\frac{\partial F}{\partial y} = g'(y) \overset{(2)}{=} y^{-1}-1 \Rightarrow g(y) = \int y^{-1}-1\,dy$$
$$= \ln|y| - y + C_2.$$

$$F(x,y) = \ln|x| + x + \ln|y| - y + C_2 = C_1$$

Soln is $\boxed{\ln|x| + x + \ln|y| - y = C}$ where $C = C_1 - C_2$

Now $y(1)=1$ $\quad \ln 1 + 1 + \ln 1 - 1 = C \Rightarrow C = 0.$

So the solution of the IVP is $\boxed{\ln|x| + x + \ln|y| - y = 0}$

Question 5. Solve the equation $(\cos^2 t - y\cos t)dt - (1+\sin t)dy = 0$.

$$\frac{dy}{dt} = \left(-y\cos t + \cos^2 t\right) / (1+\sin t)$$

$$\frac{dy}{dt} + \frac{\cos t}{1+\sin t}\, y = + \frac{\cos^2 t}{1+\sin t}$$ This is a linear eqn.

$$\mu(t) = e^{\int \frac{+\cos t}{1+\sin t} dt} = e^{+\ln(1+\sin t)} = (1+\sin t)$$ is an I.F.

$$(1+\sin t)\frac{dy}{dt} + \cos t\, y = \cos^2 t$$

$$\frac{d}{dt}\left[(1+\sin t)y\right] = \cos^2 t$$

$$(1+\sin t)y = \int\left(\frac{1}{2} + \frac{\cos 2t}{2}\right)dt = \frac{1}{2}t + \frac{1}{4}\sin 2t + c$$

$$y = \frac{1}{1+\sin t}\left(\frac{1}{2}t + \frac{1}{4}\sin 2t + c\right)$$

Question 6. The solution of the following first-order exact differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is

$$x\sin x + \cos x + xy^2 + \ln|\sec y + \tan y| = c,$$

where c is a constant. Find $M(x,y)$ and $N(x,y)$.

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0 \quad \Rightarrow \quad F = c$$

$$F(x,y) = x\sin x + \cos x + xy^2 + \ln|\sec y + \tan y| - c_1$$

$$M = \frac{\partial F}{\partial x} = \sin x + x\cos x - \sin x + y^2 = x\cos x + y^2$$

$$N = \frac{\partial F}{\partial y} = 2xy + \sec y.$$